Week 1 – Matrices and Time-State Claims

Matrices

It is easy to think of matrices as the "inside" part of tables, ie the values contained within a table without the row and column headings.

For example, if we had the table:

	Bond	Stock
Mon	54	21
Tue	55	18
Wed	56	27

The equivalent matrix would be:

$$\begin{matrix} 54 & 21 \\ 55 & 18 \\ 56 & 27 \end{matrix}$$

And to get this in MATLAB we would write:

X = [54 21; 55 18; 56 27]

Where X is the variable we want to assign the matrix to.

To recall the number in the first row and first column we would write: X(1,1)

To find the inverse of a matrix in MATLAB we type: inv(X)

Fund Allocation

If we were given the following funds allocation matrix (A):

	Fund A	Fund B	Fund C
Domestic Bonds	0.6	0.2	0.0
Domestic Stocks	0.4	0.5	0.3
Foreign Stocks	0.0	0.3	0.7

And our current portfolio (B) is:

Fund A	0.2
Fund B	0.3
Fund C	0.5

We would be able to determine our allocation among the three asset classes (X) by multiplying the matrices. That is, X = AB

If we are given a desired allocation (X), for example:

Domestic Bonds	0.15
Domestic Stocks	0.35
Foreign Stocks	0.5

And we wanted to create a portfolio (B), knowing X = AB, then if we multiply both sides by A^{-1} we get: $B = A^{-1}X$

If we wanted to do this, but had a non-square funds allocation matrix, given that we cannot invert the matrix, we cannot determine portfolios for all possible desired allocations, and we say the market is incomplete.

Time-State Claims

A numeraire is a currency or commodity.

An atomic or state-price security is a contract in which one party agrees to pay one unit of a numeraire to a second party if a particular state occurs at a particular time in the future, and pay zero numeraire in all other states. The second party will pay for the contract with present numeraire. An atomic security is known as a swap as it involves the swap of current goods for conditional future goods, and vice versa.

A zero-investment strategy swap involves trading state outcomes in the same time period, for example, party A promises to pay party B 2 numeraire in an up-state and party B to pay part A 1 numeraire in a down state. No numeraire is traded today (hence the zero-investment), and the transaction will only occur at the determined future time.

If one can trade each possible future atomic time-state claim for present units of a numeraire (and vice versa), any desired swap can be accomplished, using present units as the medium for a swap.

For example, if an atomic security for an up state was 0.3 numeraire, while it is 0.6 for a down state, one could sell one atomic security for the down state (netting them 0.6 numeraire in the present) and use this to buy two atomic securities for the up state. This would see them give up 1 numeraire if a down state occurs and gain 2 numeraire if an up state occurs. This is the equivalent of making a 2:1 up:down swap (ie the same as the initial swap example). An environment in which this kind of trade is possible is referred to as a complete market.

If using the present units as a medium results in a lower transaction price than a straight swap, an arbitrage opportunity is possible, buying using the present unit medium and selling through a swap. The opposite is also true. For example, if in the example above the up state atomic security was 0.2 numeraire, you could:

- → Sell a down state atomic security (giving you 0.6 today for the promise of giving up 1.0 in a down state)
- → Buy two up state atomic securities (paying 0.4 today for the promise of getting 2.0 in an up state)
- → Buy a 2:1 up:down swap, in which you agree to give 2.0 in an up state and receive 1.0 in a down state.

In all giving you a present day profit of 0.2.

The equivalent payment matrix is:

	Agreement	Up Atomic	Down Atomic	Net
Present	0	-0.4	+0.6	0.2
Up state	-2.0	+2.0	0	0
Down state	+1.0	0	-1.0	0

As you can see, within a payment matrix each row represents a time-state combination, while each column represents a transaction.

Arbitrage in this instance is defined as providing a positive net payoff in at least one time and state, and no negative net payoff in any time and state.

Another thing to note is for all of these examples the up and down states are the only two options (ie exhaustive) and only one can occur (ie mutually exclusive).