University of Sydney

MATH3963 LECTURE NOTES

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Contents

MATH3963 NON LINEAR DIFFERENTIAL EQUATIONS WITH APPLICATIONS (biomaths) NOTES

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Assessment:

- Midterm (Thursday April 21) 25%
- 2 assignments 10% each (due 4/4 and 30/5)
- 55% exam

Assignments are typed (eg LaTex)

PART 1: LINEAR SYSTEMS

Introduction:

- What is an ODE; how to understand them

a DE is a function relationship between a function and its derivatives

An ODE is when the function has only 1 variable independent variable.

Eg; t independant $(x(t); y(t)$ dependant

Not just scalar, but also vector valued functions (Systems of ODES)

$$\vec{x}(t) \in \mathbb{R}^d$$
$$\vec{x}(t) = \left(x_1(t), x_2(t), \dots x_d(t)\right)$$
$$x_j : \mathbb{R} \to \mathbb{R}$$

Order of ODE:

- Highest derivative appearing in equation

If you can isolate the highest derivative (eg $F\left(t, \vec{y}(t), \frac{dy(t)}{dt}, \dots \frac{d^{(n)}\vec{y}(t)}{dt^{(n)}}\right) = 0$

Explicit ODEs

If you can isolate highet derivative: it is an explicit ODE

$$\frac{d^{(n)}\vec{y}(t)}{dt^{(n)}} = G\left(t, \vec{y}(t), \frac{dy(t)}{dt}, \dots \frac{d^{(n-1)}\vec{y}(t)}{dt^{(n-1)}}\right)$$

(explicit is easier; implicit is very hard)

Notation:

Use dot

$$\frac{d}{dt} = \cdot$$

Example:

$$\dddot{y} + 2\ddot{y}y + 3(\dot{y})^2 + y = 0$$

You can reduce the order of an ODE (to 1), in exchange for making it a system of ODEs

- Introduce new variables into system

$$y_0 = y$$
$$y_1 = \dot{y}_0 = \dot{y}$$
$$y_2 = \dot{y}_1 = \ddot{y}$$

$$\therefore \dot{y}_0 = y_1$$
$$\dot{y}_1 = y_2$$
$$\dot{y}_2 = -2y_2y_0 - 3y_1^2 - y_0$$

Which is a first order system in 3 variables, equivalent to the original 3rd order ODE

Systems of ODEs

- We can play the same game for systems of ODEs

$$\ddot{\vec{y}} + A\vec{y} = 0 \quad \left(with\ \vec{y} \in \mathbb{R}^2; A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$
$$\therefore \begin{pmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0$$

$$\therefore introduce:$$
$$\dot{y}_1 = y_3$$
$$\dot{y}_2 = y_4$$
$$\dot{y}_3 = \ddot{y}_1 = -ay_1 - by_2$$
$$\dot{y}_4 = \ddot{y}_2 = -cy_1 - dy_2$$
$$\therefore \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a & -b & 0 & 0 \\ -c & -d & 0 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$
$$let\ \vec{x} = (y_1, y_2, y_3, y_4)$$

The system of equations is now:

$$\dot{\vec{x}} = B\vec{x}$$

Where $\dot{x} = \vec{f}(\vec{x})$

Example 0: $f(x) = 0$

$$\therefore \dot{x} = 0$$
$$\rightarrow \dot{x_i} = 0 \; \forall i$$
$$\vec{x} = \vec{c} \;\; for \; x, c \in \mathbb{R}^n$$

This is all the solutions to $\dot{x} = 0$; and the solutions form an n dimensional vector space

Eg 2: $f(x) = \vec{c}$ (or $\vec{f}(x) = f(t)$

$$\therefore \vec{x} = \vec{c}$$
$$\rightarrow \vec{x}(t) = \vec{c}t + \vec{d}$$

But what when $f(x)$ is a standard function

- If $f(x)$ is linear in x

Remember: "linear" if:

$$f(cx + y) = cf(x) + f(y)$$

Recall any linear map can be represented by a matrix, by choosing a basis

$$\therefore \dot{x} = A(t)x$$

If $A(t) = A$ (independent of t), it is called a constant coefficient system

$$\dot{x} = Ax$$

Eg:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$
$$\rightarrow \begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$
$$\therefore \dot{x_1} = x_2$$
$$\dot{x_2} = x_1 + x_3$$
$$\dot{x_3} = x_2 + x_4$$
$$\dot{x_4} = x_3$$

Is the system of ODEs which we now have to solve

Matrix ODEs (lecture 2)

$$\dot{x} = f(x)$$

Linear equation of x:

$$\dot{x} = Ax \quad \left(x \in \mathbb{R}^n;\ \ and\ A \in M_n(\mathbb{R})\right)$$

Eg:

$$eg: A = \begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}$$
$$\dot{x} = Ax;\ \ x = (x_1, x_2, x_3)$$
$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_! \\ x_2 \\ x_3 \end{pmatrix}$$

$$\therefore \dot{x_1} = x_2;$$
$$\dot{x_2} = x_1 + x_3$$
$$\dot{x_3} = x_2$$

So a solution is a triple vector which solves this equation

Solveing: FINDING THE EIGENVALUES/EIGENVECTORS!!!

Recall:

A value

$$\lambda \in \mathbb{C}$$

Is an eigenvalue of an $n \times n$ matrix A if a non zero vector $\vec{v} \in \mathbb{R}^n$ such that

$$Av = \lambda v$$

v is the corresponding eigenvector

$$\rightarrow (\mathrm{A} - \lambda \mathrm{I})v = 0$$
$$\rightarrow v \in \ker(A - \lambda I)$$

$$\rightarrow A - \lambda I \quad is\ not\ invertible$$
$$\rightarrow \det(A - \lambda I) = 0$$

$\det(A - \lambda I)$ is a polynomial in λ of degree n called the characteristic polynomial of A

$$\rho_A(\lambda)$$

Eigenvalues are roots of $\rho_A(\lambda) = 0$

Fundamental theorem of algebra:
$\rho_A(\lambda)$ has exactly n (possibly complex) roots counted according to algebraic multiplicity.

Algebraic multiplicity:

The algebraic multiplicity of a root r of $\rho_A(\lambda) = 0$ is the largest k such that $\rho_A(\lambda) = (\lambda - r)^k q(\lambda)$ and $q(r) \neq 0$

The algebraic multiplicity of an eigenvalue λ of A is its algebraic multiplicity as a root of $\rho_A(\lambda)$

Geometric multiplicity:

The geometric multiplicity of an eigenvalue is the maximum number of linearly independent eigenvectors

Alg multiplicity and geometric

$$algebraic\ multiplicity \geq geometric\ multiplicity$$

Returning to example:

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_! \\ x_2 \\ x_3 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = -\lambda(\lambda^2 - 2) = 0$$

$$\rightarrow \lambda = 0;\ \pm\sqrt{2}$$

Eigenvectors are found by finding the kernel of $\ker(A - \lambda I)$ for an eigenvalue:

Giving eigenvectors:

$$\lambda_1 = 0 \rightarrow\ v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\lambda_2 = \sqrt{2} \rightarrow v_2 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

$$\lambda_3 = -\sqrt{2} \rightarrow v_3 = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

Why do we care about eigenvectors/values?

$$\dot{\vec{x}} = A\vec{x}$$

Suppose that v is an eigenvector of $A \in M_n(\mathbb{C})$, with eigenvalues of λ.

Consider the vector of functions:

$$\vec{x}(t) = c(t)\vec{v}$$

c is some unkown function $c: \mathbb{C} \to \mathbb{C}$

If $\vec{x}$ solves $\dot{x} = Ax$, then

$$\dot{c}(t)\vec{v} = Ac(t)\vec{v} = c(t)A\vec{v}$$
$$= \lambda c(t)\vec{v} \ \ (as\ eigenvector)$$

As $\vec{v} \neq 0$,

$$\rightarrow \dot{c}(t) = \lambda c(t)$$

Which is a differential equation of one variable

$$\therefore c(t) = e^{\lambda t}$$

SO:

$$x(t) = e^{\lambda t}\vec{v}$$

is a solution vector to

$$\dot{x} = Ax$$

So- we have a 1D subspace of solutions $x(t) = c_0 e^{\lambda t}\vec{v}$

Notes:

1. Solutions to $\dot{x} = Ax$ are of the form $x = c_0 e^{\lambda t}\vec{v}$ where λ is a constant eigenvalue of A of $\vec{v}$ is the corresponding eigenvectors are called **EIGENSOLUTIONS**

2. What we just did has a name "ansatz" (guess what the solution could be, and see if it is correct)

Back to example:

$$\therefore x(t) = e^{0t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}; \ \ y(t) = e^{\sqrt{2}t}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}; \ \ z(t) = e^{-\sqrt{2}}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

So, as linear- any linear combination of these is the solution:

So:

$$X(t) = c_1 e^{0t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 e^{\sqrt{2}t}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} + c_3 e^{-\sqrt{2}}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

IS OUR SOLUTION‼‼!

$X(t)$ is actually a 3D vector space.

Matrix exponentials

$A \in M_{n\times n}$; what is:

$$e^A$$

$(\exp(A))$

For a matrix: DEFINE

$$e^A = \exp(A) = \sum_{(k=0)}^{\infty} \frac{1}{k!}A^k$$
$$= I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \cdots$$

If the entries converge, then: e^A IS an $n \times n$ matrix

When does e^A exist?

$\exp(A)$ first class: full set of linearly independent eigenvectors

- Hypothesis: Suppose A has a full set of linearly independent eigenvectors
- Assume that hypothesis for the rest of today(we'll look at if it's not later)

Let $v_1, \dots v_n$ be the eigenvectors; with $\lambda_1, \dots, \lambda_n$ eigenvalues

Form the following matrix:

$$P = (v_1, v_2, \dots v_n)$$

(whose columes are the eigenvectors)

$$\therefore AP = (Av_1, A\,v_2, \dots Av_n) = (\lambda_1 v_1, \dots \lambda_n v_n)\ \ (by\ multiplication)$$
$$= P\ diag(\lambda_1, \dots, \lambda_n)$$
$$:= P\Lambda$$

$$\therefore P^{-1}AP = \Lambda\ \ (= \mathrm{diag}(\lambda_1, \dots \lambda_n))$$
$$\therefore e^\Lambda = I + \Lambda + \frac{1}{2}\Lambda^2 + \cdots$$
$$= diag(e^{\lambda_1}, \dots., e^{\lambda_n})$$

LHS:

$$e^{P^{-1}AP}$$

Lemma:

$$(P^{-1}AP)^n = P^{-1}A^nP$$

So:

$$\therefore e^A = Pe^\Lambda P^{-1}\ \ \left(e^\Lambda = diag(e^{\lambda_1}, \dots, e^{\lambda_n})\right)$$

AND: the eigenvectors of $e^A = e^{\lambda_j}$, where λ_j is eigenvalues of A, *Example:*

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix};$$

$$\therefore x \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}; \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}; \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & \sqrt{2} & -\sqrt{2} \\ -1 & 1 & 1 \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} \end{pmatrix}$$

$$\therefore e^A = Pe^{\Lambda P^{-1}}$$
$$= \begin{pmatrix} 1 & 1 & 1 \\ 0 & \sqrt{2} & -\sqrt{2} \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & e^{\sqrt{2}} & 0 \\ 0 & 0 & e^{-\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & \sqrt{2} & -\sqrt{2} \\ -1 & 1 & 1 \end{pmatrix}^{-1}$$
$$= (ugly\ answer)$$

Diagonalizable or semi-simple matrix

A matrix A for which there exists an invertible matrix P such that

$$P^{-1}AP = \Lambda$$

(where Λ is diagonal is called **diagonalizable** or **semisimple**

This says, if A is semisimple, then e^A exists, and we can compute it

How big of a restriction is a full set of LI eigenvectors?

Defective matricies

If A does not have a full set of linearly independent eigenvectors, it is called **defective**

- How big of a restriction is being semisimple?
- Are there many semi simple matrices?

Proposition:

Suppose $\lambda_1 \neq \lambda_2$ are eigenvalues of a matrix A with eigenvectors $v_1\ and\ v_2$. Then v_1 and v_2 are linearly independent

Proof:

If ∃ a dependence relation, then $\exists k_1 \ and\ k_2$ scalars such that

$$k_1 v_1 + k_2 v_2 = 0 \quad (k_1, k_2 \neq 0)$$

Applying A:

$$A(k_1 v_1 + k_2 v_2) = 0$$
$$Ak_1 v_1 + Ak_2 v_2 = 0$$
$$\therefore k_1 \lambda v_1 + k_2 \lambda_2 v_2 = 0 \quad (as\ eigenvectors)$$
$$\rightarrow take - \lambda_1 k_1 v_1 - \lambda_1 k_2 v_2 = 0$$
$$\backslash \therefore (\lambda_2 - \lambda_1) k_2 v_2 = 0$$
$$\rightarrow k_2 = 0 \quad (\therefore contradiction)$$

Example:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
$$\rho_A(\lambda) = \lambda^2 - (a+d)\lambda + ad - bc$$
$$= \lambda^2 - \tau\lambda + \delta \ \ (taking\ \tau = a + d; \delta = ad - bc)$$
$$\left(\tau = Trace(A) = \sum_{j=1}^{n} a_{jj} \quad (sum\ of\ the\ diagonals) \right)$$
$$\therefore \lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2}\ are\ roots$$
$$\therefore \lambda_+ \neq \lambda_-\ if\ \tau^2 - 4\delta = 0$$

$$\therefore \tau^2 - 4\delta = 0$$
$$\rightarrow \delta = \frac{\tau^2}{4}$$

Therefore: any matrix which does not lie on this parabola in τ, δ space will be diagonalizable

$$\tau^2 - 4\delta = (a-d)^2 + 4bc$$

The set of matricies with repeated eigenvalues, is the locus of points with

$$(a-d)^2 + 4bc = 0$$

A 3D hypersurface, in 4D space

So what if A is defective? Can we compute e^A

- Yes

Two $n \times n$ matricies A, B are commutative, if

$$AB = BA$$

Commutator

A commutator we will define as:

$$[A,B] \coloneqq AB - BA = 0 \ \ (iff\ AB\ commute)$$

Proposition

If A and B commute, then

$$e^{A+B} = e^A + e^B$$

- Aside: if $e^{A+B} = e^A e^B$, and A and B satisfy symmetric → $AB = BA$

Proof

$$e^{A+B} = I + (A+B) + \frac{1}{2}(A+B)^2$$
$$= I + A + B + \frac{1}{2}(A^2 + AB + BA + B^2) + \cdots$$
$$= I + A + B + \frac{1}{2}A^2 + AB + \frac{1}{2}B^2 \ \ (as\ commutative)$$

And:

$$e^A e^B = \left(I + A + \frac{1}{2}A^2 \ldots\right)\left(I + B + \frac{1}{2}B^2 \ldots\right)$$
$$= I + (A+B) + \frac{1}{2}A^2 + AB + \frac{1}{2}B^2 + \cdots$$

Example:

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \ (a \neq b); B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$\therefore e^A = \begin{pmatrix} e^a & 0 \\ 0 & e^b \end{pmatrix}; e^B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
$$e^{A+B} = \begin{pmatrix} e^a & \dfrac{e^a - e^b}{a-b} \\ 0 & e^b \end{pmatrix}$$
$$e^A e^B = e^{A+B} \rightarrow (a-b)e^a = e^a - e^b$$

If $x = a - b$

$$\therefore x = 1 - e^{-x}$$

One example of a solution: is

$$x \approx -6.6022 - 736.693\, i$$

Nilpotent matrix:

Definition:

A matrix N is nilpotent if some power of it is $= 0$ ($N^T = 0$ for some T)

Eg; $N = \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix} \rightarrow N^3 = 0 \ \ (calculate)$

This means that the power series terminates for e^N :

$$e^N = I + N + \frac{1}{2}N^2 + 0 + 0 + \cdots$$

So- how do we compute the exponential?

Jordan Canonical Form:

Let A be an $n \times n$ matrix, with complex entries $\exists$ a basis of $\mathbb{C}^n$ and an invertible matrix B (of basis vectors) such that

$$B^{-1}AB = S + N$$

Where S is diagonal (semisimple); and N is nilpotent; and S and N commute $(SN - NS = 0)$

So; this means that e^A exists for all matricies:

$$B^{-1}AB = S + N$$
$$\to B^{-1}e^A B = e^S e^N$$

$$e^A = Be^S e^N B^{-1}$$

This is cool.

Matrix and other term exponentials:

So now: consider the follow At; where $A \in M_n(\mathbb{R})$ and $t \in \mathbb{R}$

So:

$$e^{At} = \sum_{k=0}^{\infty} \left(\frac{t^k}{k!}\right) A^k$$

For a fixed A; the exponential is a function map between real numbers and matricies

$$e^{At}: \mathbb{R} \to M_n(\mathbb{R})$$

$$\exp_A(t) : t \to e^{At}$$
$$\exp_A(t+s) = e^{A(t+s)} = e^{At}e^{As} = \exp_A(t)\ \exp_A(s)$$

e^{At} is invertible! Inverse is e^{-At}

So- what is the derivative between the map of the real numbers to the matricies?

Complex numbers and matricies:

Let $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$\to J^2 = -I;\ \ J^2 = -J; J^4 = I$$

$$e^{Jt} = I + Jt + \frac{1}{2}J^2t^2 + \cdots$$

$$= I + Jt - \frac{1}{2}t^2 I - \frac{1}{3!}t^3 J$$
$$= I - \frac{t^2}{2}I + \frac{t^4}{4!} + \cdots + J\left(t - \frac{t^3}{3!} + \cdots\right)$$
$$= \cos t\, I + \sin t\, J$$
$$= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

Which is the rotation matrix;

So: the matrix J is the complex number i; and we wrote Euler's formula

$$e^{it} = \cos t + i \sin t$$

- We can actually take any complex number $z = x + iy$; and find its corresponding matrix

$$M_z \coloneqq xI + yJ = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

(the technical term is a field isomorphism)

Complex number algebra:

$$M_{z_1+z_2} = M_{z_1} + M_{z_2}$$
$$M_{z_1 z_2} = M_{z_1} M_{z_2} = M_{z_2} M_{z_1} = M_{z_2 z_1}$$

Complex exponentials:

$$M_{e^z} = e^{Mt}$$

$$z = x + iy:$$
$$e^z = e^x \cos y + ie^x \sin y = e^x \begin{pmatrix} \cos y & -\sin y \\ \sin y & \cos y \end{pmatrix}$$
$$= e^x I + e^y J$$

Complex number things:

$$\det(M_z) = x^2 + y^2 = |z|^2$$
$$M_{\bar{z}} = \begin{pmatrix} x & y \\ -y & x \end{pmatrix} = M_z^T$$
$$M_r = rI \quad (if\ r \in \mathbb{R})$$
$$M_{\bar{z}} M_z = M_{|z|^2} = |z|^2 I$$
$$if\ z \neq 0; M_z\ is\ invertible; and$$
$$M_z^{-1} = \frac{1}{|z|^2} M_z^T = \frac{1}{\det(M_z)} M_Z^T$$

So lets compute $e^{Jt} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$:

Eigenvalues of Jt are $\pm it$

e –vectors are $\begin{pmatrix} \pm i \\ 1 \end{pmatrix}$

$$P = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}; \; P^{-1} = \frac{1}{2i}\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}$$
$$\Lambda = \begin{pmatrix} it & 0 \\ 0 & -it \end{pmatrix}$$
$$\to e^{Jt} = \frac{1}{2i}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}$$
$$= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

So we got what we started with! (which is good)

Real normal form of matricies

Moving towards a real "normal" form for matricesl with real entries but complex eigenvalues

Fact: if $A \in M_n(\mathbb{R})$ then if λ is an eigenvalue; so $\bar{\lambda}$ is an e-value. (complex conjugate theorem). Also- eigenvectors come in conjugate pairs as well

$$\overline{Av = \lambda v} \to A\bar{v} = \bar{\lambda}\bar{v}$$

- If λ is an eigenvalue $\lambda = a + ib$; then $\bar{\lambda} = a - ib$ is also an e-value
- If $\vec{v} = \vec{u} + i\vec{w}$; $(u, w \in \mathbb{R}^n) \to \bar{\vec{v}} = \vec{u} - i\vec{w}$

So:

$$2u = v + \bar{v}$$
$$A(2u) = \lambda v + \lambda\bar{v} = 2(a\vec{u} - b\vec{w})$$
$$\to A\vec{u} = (a\vec{u} - b\vec{w})$$

Similarly;

$$A\vec{w} = (a\vec{u} + b\vec{w})$$

So; if P is the $n \times 2$ matrix

$$P = (u \quad w)$$
$$\to AP = P\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

(we are close to finding the algorithm for computing the matrix exponential of any matrix)

Eg:

$$A = \begin{matrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{matrix}$$

Characteristic polynomial is:

$$x^4 + 3x^2 + 1$$

e-values of A are $\pm i\phi \;\; and \pm \frac{i}{\phi}$ (with $\phi = \frac{1+\sqrt{5}}{2}$ the golden ratio) so eigenvectors:

$$v_1 = \begin{pmatrix} i \\ -\phi \\ -i\phi \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -\phi \\ 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \\ -\phi \\ 1 \end{pmatrix} = u_1 + iw_i$$

$$; \ \ v_2 = \begin{pmatrix} i\phi \\ -1 \\ i \\ -\phi \end{pmatrix}$$

$$\rightarrow$$

So

$$AP_1 = P_1 \begin{pmatrix} 0 & -\phi \\ \phi & 0 \end{pmatrix}$$

$$P = (u_1 w_1 u_2 w_2) P^{-1}AP = \begin{pmatrix} 0 & -\phi & 0 & 0 \\ \phi & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\phi} \\ 0 & 0 & \frac{1}{\phi} & 0 \end{pmatrix}$$

this is called a block diagonal matrix (has blocks of diagonal matricies; and 0s everywhere else)

$$= \phi J \oplus -\frac{1}{\phi} J$$

Direct sum:

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

A and B don't need to be the same size

Lemma: if

$$A = B \oplus C$$

Then

$$e^A = e^B \oplus e^C$$

Proof:

$$\left(A = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} \right) \text{ which commute}$$

$$\rightarrow e^A = e^{\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}} e^{\begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}} = \begin{pmatrix} e^B & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & e^C \end{pmatrix} = \begin{pmatrix} e^B & 0 \\ 0 & e^C \end{pmatrix} = e^B \oplus e^C$$

Repeated eigenvalues

So what do we do if we have repeated eigenvalues? How do we calculate the matrix exponential?

Generalized eigenspace:

Definition

Suppose λ_j is an eigenvalue of a matrix A with algebraic multiplicity n_j. We define the generalized eigenspace of λ_j as:

$$E_j := \ker\left[\left(A - \lambda_j I\right)^{n_j}\right]$$

There are basically two reasons why this definition is important. The first is that these generalized eigenspaces are INVARIANT UNDER MULTIPLICATION BY A (does not change). That is: if $\boldsymbol{v} \in E_j$, then $A\boldsymbol{v}$ is as well

Proposition: of generalized eigenspace

Suppose that E_j is a generalized eigenspace of the matrix A, then if $v \in E_j$; so is Av

Proof of proposition:

- If $v \in E_j$ for some generalized eigenspace, for some eigenvector λ; then $v \in \ker\left((A - \lambda I)^k\right)$ say for some k. Then we have that $(A - \lambda I)^k \vec{v} = (A - \lambda I)^k (A - \lambda I + \lambda I) v$
$$= (A - \lambda I)^{k+1} + (A - \lambda I)^k \lambda v = 0$$

The second reason we care about this is that it will give us our set of linearly independent generalized eigenvectors; which we can use to find e^A

Primary decomposition theorym

Let $T: V \to V$ be a linear transformation of an n dimensional vector space over the complex numbers. If $\lambda_1, \dots \lambda_k$ are the distinct eigenvalues (not necessarily $= n$) ; if $\mathrm{E_j}$ is the eigenspace for λ_j, then $\dim\left(E_j\right) =$ the algebraic multiplicity of λ_j and the generalized eigenvectors span V. In terms of vector spaces, we have:

$$V = E_1 \oplus E_2 \oplus \dots \oplus E_k$$

Combining this with the previous proposition: T will decompose the vector space into the direct sum of the invariant subspaces.

The next thing to do is explicitly determine the ssemisimple nilpotent decomposition. Before: we had the diagonal matrix Λ which was

- Block diagonal
- In normal form for complex eigenvalues

- Diagonal for real eigenvalues

Suppose we have a real matrix A with n eigenvalues $\lambda_1, \dots\ \lambda_n$ with repeated multiplicity, suppose we break them up into complex ones and real ones, so we have $\lambda_1, \overline{\lambda_1}, \dots, \lambda_k, \overline{\lambda_k}$ complex and $\lambda_{2k+1}, \dots \lambda_n$ real. Let $\lambda_j = a_j + ib + j$ then:

$$\Lambda := \begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix} \oplus \dots \oplus \begin{pmatrix} a_k & b_k \\ -b_k & a_k \end{pmatrix} \oplus diag(\lambda_{2k+1}, \dots \lambda_n)$$

Let $v_1, \overline{v_1}, \dots v_{2k+1}, \dots, v_n$ be a basis of $\mathbb{R}^n$ of generalsed eigenvectors, then for complex ones: write $v_j = u_j + iw_j$

Let

$$P := (u_1\ w_1\ u_2\ w_2 \dots u_k\ w_k\ v_{2k+1} \dots v_n)$$

Returning to our construction, the matrix P is invertiable (because of the primary decomposition theorem); so we can write the new matrix as

$$S = P\Lambda P^{-1}$$

And a matrix

$$N = A - S$$

We have the following:

Theorems:

Let A, N S Λ and P be the matricies defined: then

1. $A = S + N$
2. S is semisimple
3. S, N and A commute
4. N is nilpotent

Proofs of theorems:

1. From definition of N
2. From construction of S
3. $[S, A] = 0$ suppose that v is a generalized eigenvector of A associated to eigenvalue λ. By construction: v is a genuine eigenvector of S. Note that eigenspace is invariant, $[S, A]v = SAv - ASv = \lambda(Av - Av) = 0$ so N and S commute. A and N commute from the definition of N and that A commutes with S
4. suppose maximum algebraic multiplicity of an eigenvalue of A is m. Then for any $v \in E_j$ a generalized eigenspace corresponding to the eigenvalue λ_j. We have $N^m v = (A - S)^m v = (A - S)^{m-1} v(A - \lambda_j)v = (A - \lambda_j)^m v = 0$

General algorithm for finding e^A

1. find eigenvalues

2. find generalized eigenspace for each eigenvalue.; eigenvalues Λ
3. Turn these vectors into P
4. $S = P\Lambda P^{-1}$
5. $N = A - S$
6. $e^{At} = e^{St}e^{Nt} = Pe^{\Lambda P^{-1}}e^{(A-S)t}$

Example:

$$\begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix}$$

Eigenvalues*:*

$$\det\left(\begin{pmatrix} -1-\lambda & 1 \\ -4 & 3-\lambda \end{pmatrix}\right) = 0$$

$$\rightarrow \lambda = 1{,}1$$

$$\Lambda = diag(1{,}1) = I$$

Generalized eigenspace:

$$\ker((A - \lambda I)^{n_j}) = \ker((A - I)^2) = \ker\begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix}^2 = \ker\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) = \mathbb{R}^2$$

So our vectors must span $\mathbb{R}^n$; choose

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix};\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\rightarrow S = I\,II^{-1} = I$$

$$\rightarrow N = A - S = A - I = \begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix}$$

So then:

$$e^A = e^{ST}e^{Nt}$$

Last time:

Algorithm for computing e^A and e^{At}

Eg: let

$$A = \begin{pmatrix} 1 & 0 & 0 & 2 \\ -2 & -2 & 3 & 4 \\ -1 & -2 & 3 & 2 \\ -1 & -2 & 1 & 4 \end{pmatrix}$$

Evals ar $\lambda = 2,2,1,1$

Generalized eigenspace for $\lambda = 2$

$$E_2 = \ker((A-2I)^2)$$

E_2 is spanned by:

$$\begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}; \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$E_1 = \ker((A-I)^2)$ spanned by

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\Lambda = diag(2,2,1,1); \;\; S = P\Lambda P^{-1}$$
$$P = \begin{pmatrix} 2 & -2 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$
$$P^{-1} = \cdots$$
$$S = P\Lambda P^{-1}$$

$$N = AS$$

Note: if not using a computer:

Check that N is nilpotent ($N^a = 0$ for some $a\ at\ most\ the\ size\ of\ the\ matrix,$)
check $NS - SN = 0$ (that they commute)

$$\therefore At = (S+N)t = St + Nt$$
$$\rightarrow e^{At} = e^{St}e^{Nt} = Pe^{\Lambda t}P^{-1}e^{Nt}$$
$$= Pe^{\Lambda t}P^{-1}(1 + Nt)$$

$$e^{At}: \mathbb{R} \to GL(n, \mathbb{R}) = (Space\ of\ invertible\ n \times\ n\ matricies\ with\ real\ entries)$$

Derivative of e^{At}

Proposition: what is

$$\frac{d}{dt}(e^{At}) = Ae^{At} = e^{At}A$$

$$(Ae^{At} = e^{At}A\ from\ that\ e^{At}\ exists\ and = power\ series\ representation)$$

Proof 1:

$$\frac{d}{dt}(e^{At}) = \frac{d}{dt}\left(I + tA + \frac{1}{2}t^2A^2 + \frac{t^3}{3!}A^3 + \cdots\right)$$
$$= A + tA^2 + \frac{t^2}{2}A^3 + \frac{t^3}{3!}A^4 + \cdots$$
$$= A\left(1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \cdots\right)$$

(we will show it converges uniformly, and so we can differentiate it term by term)

$$= Ae^{At}$$

Proof 2:

$$\frac{d}{dt}(e^{At}) = \lim_{h\to 0}\left(\frac{e^{A(t+h)} - e^{At}}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{e^{At}\left(e^{Ah} - I\right)}{h}\right)$$
$$= e^{At}\lim_{h\to 0}\left(\frac{\left(Ah + \frac{A^2h^2}{2} + \cdots\right)}{h}\right)$$
$$= e^{At}\lim_{h\to 0}\left(A + \frac{A^2}{2}h + \cdots\right)$$
$$= e^{At}A$$

This proposition implies that e^{At} is a solution to the matrix differential equation:

$$\dot{\Phi}(t) = A\Phi(t)$$

$$(note: A\ is\ constant\ coefficient)$$

A cannot be functions of t, as the functions may not commute with the integral

Comparing this to the 1x1 differential equation:

$$\dot{y} = ay$$
$$\to y = Ce^{at}$$