STAT2911 Lecture Notes

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Contents

PROBABILITY AND STATISTICAL MODELS STAT2911 Notes

Table of discrete distributions

Distribution	Model	pmf	$E(X)$	$V(X)$
$Bernoulli(p)$	Success or failure, with probability p Eg: coin toss	p	p	$p(1-p)$
$Geometric(p)$	Probability p to first success: eg coin toss till first head	$(1-p)^{k-1}p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
$Binomial(n,p)$	n Bernoullii trials. Eg n coin tosses	$\binom{n}{k}p^k(1-p)^{n-k}$	np	$np(1-p)$
$Poisson(\lambda)$	Number of events in a certain time interval. Eg: number of radiactive particles emmited in certain time	$e^{-\lambda}\frac{\lambda^k}{k!}$	λ	λ
$Hypergeometric$ (r,n,m)	number of red balls r in a sample of m balls drawn **without** replacement from an urn with r red balls and $n-r$ black balls	$\frac{\binom{r}{k}\binom{n-r}{m-k}}{\binom{n}{m}}$	$\frac{mr}{n}$	$m\frac{r}{n}\frac{n-r}{n}\frac{n-m}{n-1}$
$Negative$ $binomial(r,p)$	Number of Bernoulli trials till the r^{th} success	$\binom{k-1}{r-1}(1-p)^{k-r}p^r$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$

Uri Keich; Carslaw 821; Monday 5-6

Introduction

- Probability in general and this course has 2 components:
    - Mathematical theory of probability
        - Definitions, theorems and proofs
        - Abstraction of experiments whose outcome is random
    - Modelling: argued rather than proved applications can be confusing, as well as ill defined
        - Useful for describing/summarising the data and making accurate predictions

Mathematical theory of probability

Sample space

The set of all possible outcomes, denoted Ω, is the sample space

Sample point

A point $\omega \in \Omega$ is a sample point

Examples:

Die:

$$\Omega = \{1,2,3,4,5,6\}$$

Coin:

$$\Omega = \{H, T\}$$

- Complication: do we model the possibility that the coin can land on its side?

Events:

- Events are subsets of Ω, for which we can assign a probability.
- We say an event A occurred if the outcome, or sample point $\omega \in \Omega$ satisfies $\omega \in A$

Probability as a measure

$P(A)$ is the probability of the event A, which **intuitively** is the rate at which A occurs if we repeat the experiment many times.

Mathematically: P is a probability measure function if:

1. $P(\Omega) = 1$
2. $P(A) \geq 0$ for any event A
3. If $A_1, A_2, \ldots$ are mutually disjoint $(A_1 \cap A_2 \cap \ldots = \emptyset)$ then $(P(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n))$

For this to make sense, we need to know that a union of a sequence of events is ALSO an event.

- Why do we bother with determining events? Why can't any subset of Ω be an event?
    - Imagine choosing a point at random in a large cube $C \subset \mathbb{R}^3$, where the probability of the point lying in any set $A \subset C$ is proportional to its volume $|A|$
    - Clearly, if $A, B \in C$ are related through a rigid motion (translation + rotation), then $|A| = |B|$ so their probabilities are the same
    - Similarly, if A ∈ C can be split into $A_1 \cup A_2$, then $P(A) = P(A_1) + P(A_2)$
    - Bernard-Tarski Paradox:
        - The unit ball B in $\mathbb{R}^3$ can be decomposed into 5 pieces, which can be assembled using only rigid motions into two balls of the same size.
        - But then: $P(B) = P(B_1) + P(B_2) + \cdots + P(B_5) = P(B_1') + \cdots + P(B_5') = 2P(B)$
        - This is why we cannot assign probabilities to EVERY subset of Ω. Probabilities can't be assigned to arbitrary sets, rather to sets which are "measurable".

The collection of measurable sets is captured by the notion of σ algebra (σ –field)

Sigma algebra

Definition:

A collection of subsets of Ω is a $\sigma - algebra$ is

1. $\Omega \in F$
2. $A \in F \Longrightarrow A^c \in F$
3. $A_1, A_2, \ldots \in F \Longrightarrow \cup_{n=1}^{\infty} A_n \in F$

2\. says that F is closed with respect to complementing and taking the complement

3\. says that F is closed with respect to a countable union

- (countable means you can count it)

    $A = \{1,3,5\}$ is a finite countable set

- $\mathbb{N}$ is an infinite countable set
- $\mathbb{Z}$ is also an infinite countable set
- $\mathbb{Q}$ is infinite countable
- $\mathbb{R}$ is not countable

Example of sigma algebra

$$\Omega = \{1,2,\ldots 6\}$$

F = the power set of Ω = the set of all subsets of Ω

$$= \{\emptyset, \{all\ singles\}, \{all\ doubles\} \ldots, \Omega$$

Denoted

$$= 2^{\Omega}$$

Questions:

What is the cardinality of F, or how many subsets of Ω does it contain

-

Is 2^{Ω} ALWAYS a σ algebra

What is the smallest σ algebra regardless of the sample space

ANSWERS?

- $2^{|\Omega|}$?

- $\{\emptyset, \Omega\}$

Probability space:

A probability space consists of

1. A sample space Ω
2. A $\sigma -$ algebra of subsets of Ω, F
3. A probability measure: $P: F \to \mathbb{R}$

Eg: model of fair dice

$$\Omega = \{1,2,\ldots 6\}$$

$$F = 2^{\Omega}$$

$$P(\{,\}) = \frac{1}{6}, \qquad i = [1,6]$$

$$P(A) = \frac{|A|}{6}\ (for\ A \in F)$$

Venn diagrams:

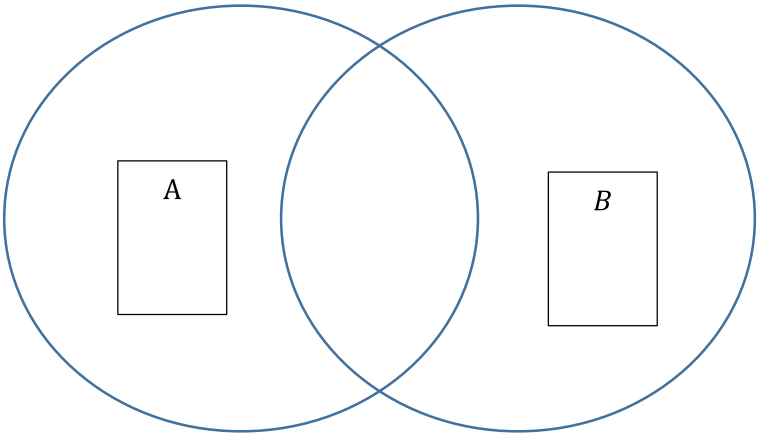
Useful to vizualise relations between sets. They help plan proofs, but don't use them as PROOFS in a course

De morgan's laws (for set theory):

$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

Eg:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



- Why is $A \cup B$ and $\cap B \in F$? (why are they events)

$$A \cup B = A \,\dot{\cup}\, (B \setminus A) \quad [(\dot{\cup}) = disjoint\ union)$$
$$B \setminus A = B \cap A^c$$

$$\therefore P(A \cup B) = P(A) + P(B \setminus A)$$
$$\left(using\ that\ (P(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n), \qquad with\ A\ and\ B\ and\ infinite\ \emptyset\ 's \right)$$

(probability of empty set is 0)

$$P(\emptyset)P(\cup\,\emptyset) = \sum_{n=1}^{\infty} P(\emptyset) \to P(\emptyset\ ) = 0$$

$$P(B) = P(B \setminus A) + P(B \cap A)$$
$$\to P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Equiprobable spaces

- A space consistics of a finite sample space such that $\forall \omega \in \Omega\ \ P(\{\omega\}) = c > 0$
- (all equally likely)

$$NOte:\ \omega \in \Omega\, is\ a\ sample\ point, \qquad \{\omega\} = an\ event\ in\ F$$

- Why does it have to be finite?
  - Otherwise

$$\{\omega\}_{n=1}^{\infty} \subset \Omega$$
$$P(\cup_n \{\omega_n\}) = \sum_n P(\omega_n) = \sum_{n=1}^{\infty} c = \infty$$
$$\left[but\ \forall A \subset F, P(A) \in [0,1]\ \ (why? \to show)\right)$$

Claim: $P(A) = \frac{|A|}{|\Omega|}$

Proof:

$$A = \cup_{\omega \in A} \{\omega\}$$
$$\therefore P(A) = P(\cup_{\omega \in A} \{\omega\}) = \sum_{\omega \in A} P(\omega)$$
$$= |A| \times c$$
$$take\ A = \Omega$$
$$\to P(\Omega) = 1\ (definition) = |\Omega| c$$
$$\to c = \frac{1}{|\Omega|}$$
$$\to P(A) = \frac{|A|}{|\Omega|}$$

This means: that probability is just combinatorics!!!!

Example of probability combinatorics

1. A fair die is rolled:

$$P(\{\omega\}) = \frac{1}{6}$$

2. A group of n people meet at a party, what is the probability that at least 2 od them share a birthday
   - model that no leap years, no association between birthdays

$$\Omega = \{(i_1, \dots i_n) \big| i_k \in \{1, \dots 365\}\}$$
$$\to \frac{1}{|\Omega|} = 365^{-n}$$
$$A = \{(i_1, \dots i_n) \in \Omega\} \exists j \neq k, i_j = i_k$$
$$Let's\ rather\ compute\ the\ complement$$
$$A^c = \{(i_1, \dots i_n) \in \Omega\} \exists j \neq k, i_j \neq i_k$$

$$P(A^c) = \frac{|A^C|}{|\Omega|} = \frac{365 \times 364 \times \ldots (365 - n + 1)}{365^n}$$
$$= \prod_{i=1}^{n} \frac{365 - (i - 1)}{365} \quad \{n \leq 365\}$$
$$= \prod_{i=1}^{n} 1 - \frac{i - 1}{365}$$
$$P(A) = 1 - \frac{\binom{365}{n}}{365^n}$$

Sidenote: for $n = 23 \rightarrow P(A) \approx \frac{1}{2}$

3. what is the probability that one of the guests shares YOUR birthday

$$B = \{(i_1, \ldots i_n) \in \Omega\} \exists j, i_j = x \ \ (your\ birthday)\}$$

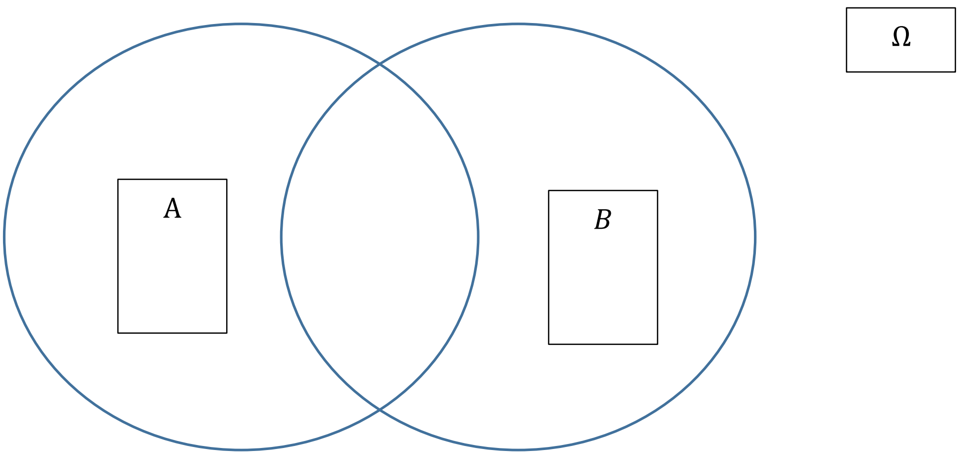
$$\therefore B^c = \{(i_1, \ldots i_n) \in \Omega\} \forall j, i_j \neq x$$
$$= |B^c| = 364^n$$
$$\rightarrow P(B^c) = \frac{364^n}{365^n}$$
$$= \left(1 - \frac{1}{365}\right)^n \approx 3^{-\frac{n}{365}}$$

Sidenote: for $n = 253, P(B) \approx \frac{1}{2}$

Conditional probability:

If $A, B \in F$ and $P(B) > 0$ we can define (Probability of A given B)

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$



Example:

$$\Omega = \{1, \ldots, 6\}$$

$$A = \{1,2\};\ \ B = \{1,3,5\}$$

Independence

Event $A \in F$ is independent (ind) of $B \in F$ if knowing whether or not A occurred, does **not** give us any information on whether or not B occurred.

$$\therefore P(B|A) = P(B)$$

$\rightarrow \frac{P(B \cap A)}{P(A)} = P(B)$

$$\therefore P(B \cap A) = P(A)P(B)$$

- this definition is more robust (symmetric about A and B), and no need to assume $P(A) > 0$; and easier to generalise

general:

$$P(A_1, \cap A_2, \ldots \cap A_n) = P\left(\prod_{i=1}^{n} A_i\right)$$

- stronger than pairwise independence‼!

NOTE: independence is NOT DISJOINT (disjoint gives you total knowledge that the other would not have occurred)

Law of total probability

- a probability of an event may be computed by summing over all eventualities

3 machines	A	B	C
Production rate	.05	.2	.3
Failure rate	.01	.02	.005

$$P(failed\ product) = .01(.05) + .2(.02) + .3(.005)$$

Generally:

If $P_j \in F$ forms a partition of Ω

$$\Omega = \dot{\cup}_j B_j$$
$$then\ P(A) = P(A \cap \Omega)$$
$$= P\left(A \cup \left(\dot{\cup}_j B_j\right)\right) = P\left(\dot{\cup}_j \left(A \cap B_j\right)\right) = \sum_j P(A \cap B) = \sum_j P(A)P(B|A)$$

Distributive law for set theory:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Baye's rule:

Diagnostic: which of the events B_j triggered the event A?

$$P(B_j|A) = \frac{P(A \cap B_j)}{P(A)}$$

(which machine B caused the failure of A?)

$$= \frac{P(A|B_j)P(B_j)}{\sum_i P(A|B_i)P(B_i)}$$
$$(using\ definitions\ of\ law\ of\ total\ probability\ and\ conditional\ probability$$

Random Variables

A RV is a measurable function $X: \Omega \to \mathbb{R}$

Discrete RV

A RV is is discrete if its range:

$$X(\Omega) = \{X(\omega): \omega \in \Omega\} \subset \mathbb{R}$$

Is a countable set (finite or infinite)

Probability mass function of Discrete RV:

The pmf pf X is definite as:

$$p_X(x) = P(X = x) = P(\{\omega: X(\omega) = x\}) \quad [with\ x \in \mathbb{R}]$$

- why is $\{\omega: X(\omega) = x\} \in F$?

Properties of pmf

Claim:

1. $\forall x \in \mathbb{R} \setminus X(\Omega)$ then $p_X(x) = 0$ (the pmf of something which is unattainable is 0)
2. With $\{x_i\} = X(\Omega)$, $\sum_i p_X(x) = 1$ (the pmf of all outcomes is 1)

The distribution of a discrete RV is completely specified by its pmf. Indeed, $\forall A \subset \mathbb{R}$

$$P(X \in A) = \sum_{i: x_i \in A} p_X(x_i)$$

(shows the probability that an outcome is going to happen)

- We can thus specify the distribution of a RV X by specifying its pmf p_X

Question:
Can any functions $p: \mathbb{R} \to [0,1]$ with a countable support $\{x: p(x) > 0\}$ such that $\sum_{i:p(r)>0} p(x_i) = 1$ be a pmf for some random variable?

ClaimL if p is as above, then there exist a probability space (Ω, F, p) obout a RV $X: \Omega \to \mathbb{R}$ such that $p_X = p$

Proof: $\Omega = \{x: p(x) > 0\}; F = 2^{\Omega};\ \ P(A) = \sum_{x \in A} p(x)$

$$X(\omega) = \omega$$

Examples of random variables

Bernoulli random variable

Is defined by the pmf

$$p(x) = \begin{cases} 1-p; & x = 0 \\ p; & x = 1 \end{cases} \quad (failuer\ and\ success)$$

Eg:

$$\Omega = \{H, T\}:\ \ P(H) = p$$
$$X(\omega) = \begin{cases} 1; & \omega = H \\ 0; & \omega = T \end{cases}$$

Binomial random variable

S_n models the number A of some in an iid (independatn and identically distributed) Bernoulli trials

A 2 parameter family of distributions: (n, p)

Note: if $X_i = 1$: $\{success\ of\ trial\} = \begin{cases} 0 & if\ failure \\ 1 & of\ success \end{cases}$; then $X_i \sim Bernoulli(p)$ and $S_n = \sum_1^n X_i$

Generally: fpr an event A, then the random variable 1_A is the indicator function of A:

$$1_A(RV) = \begin{cases} 1 & if\ \omega \in A \\ 0 & \omega\ \ not\ in\ A \end{cases}$$

$$S_n(\Omega) = \{0,1,2, \dots n\}$$

For $k \in S_n(\Omega)$; $p(S_n = k)$ is:

- Consider the configuration of a Bernoulli trials: $s, s, s, \dots . f, f, f..$
- Its probability is $p^k(1-p)^k \times \binom{n}{k}$

Binomial RV pmf

$$\therefore p(S_n = k) = \binom{n}{k} p^k (1-p)^k$$

Geometric random variable:

Models the number of iid Bernoulli (p) trials if it takes till the first success

$X(\Omega) = \{1,2,3 \dots\} \cup \{\infty\}$

A 1-parameter family : p

Pmf of geometric random variable

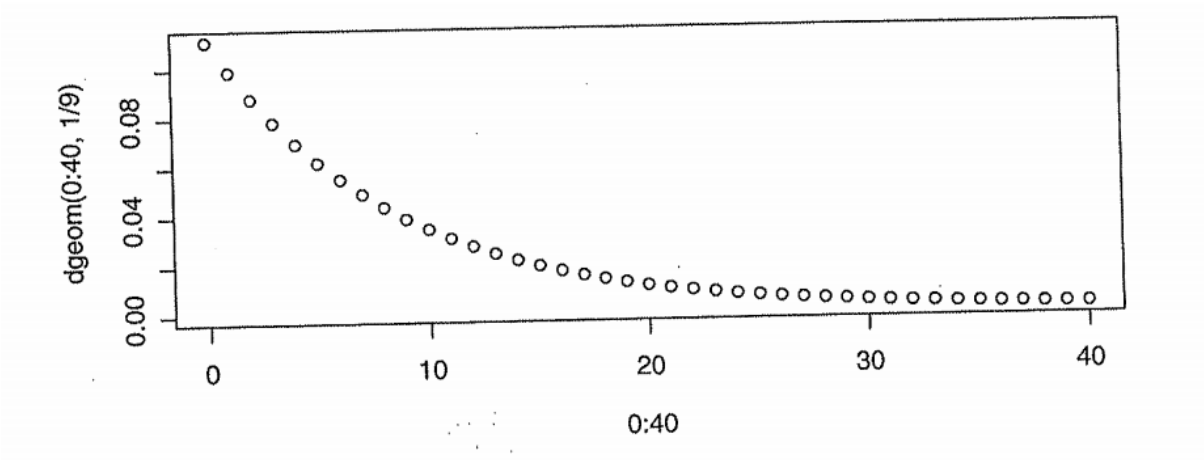
$$p(X = k) = (1-p)^{k-1}p$$

$$P(X > k) = (1-p)^k$$
$$P(X > n + k | X > k) = \frac{P(X > n + k, X > k)}{P(X > k)} \quad (using\ conditional\ probability)$$
$$= \frac{P(X > n + k)}{P(X > k)} = \frac{(1-p)^{n+k}}{(1-p)^k} = (1-p)^n = P(X > n)$$

This property is "memoryless" (the probability does not remember what has happened before)

- Geometric is the ONLY discrete memoryless distribution

If $x_i = 1_{\{succsess\ of\ trial\ i\}}$, then $x = \min\{i: X_i = 1\}$



Negative binomial RV

Models the number of iid Bernoulli trials till the r^{th} success, where $r \in \mathbb{N}$

- A 2 parameter disctribution
- Note: $X^1 : r = 1$ is a geometric random variable

$$X^r(\Omega) = \{r, r + 1, \dots\} \cup \{\infty\}$$

Pmf:

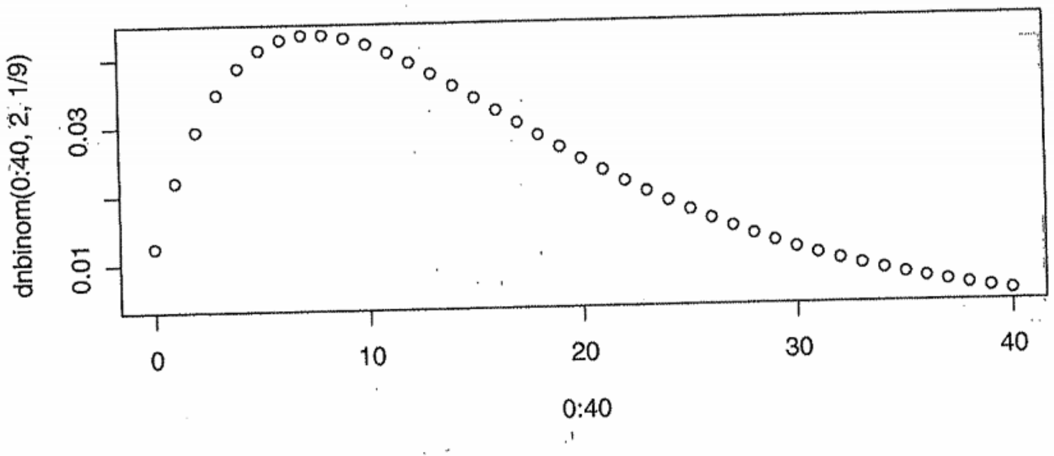
For $k \in X^r(\Omega), P(X^r = k) =$

- Consider the configuration of its Bernoulli trials

$$(f, f, f, \dots . s, s, s, s)$$

$$p = \binom{k-1}{r-1}(1-p)^{k-r}p^r$$

$$X^r = \min\left\{m: \sum_1^n x_i = r\right\}$$



Hypergeometric RV

X models the number of red balls in a sample of m balls drawn **without** replacement from an urn with r red balls and $n - r$ black balls

$$X(\Omega) \subset \{0,1, \dots r\}$$

Probability:

For $k \in X(\Omega)$

$$pmf: \; P(X = k) = \frac{\binom{r}{k}\binom{n-r}{m-k}}{\binom{n}{m}}$$

Which is a 3 parater family: (r, n, m)

(is not phrased in iid Bernoulii RV)

- A famous case of this is the Fisher Exact Test

Fisher Exact Test:

30 convicted criminals with same sex twin of 13 which 13 were identical and 17 were non identical twins.

- Is there evidence of a genetic link?

-

	Convicted	Not convicted	
Identical	10	3	13
different	2	15	17
	12	18	

- Assuming that wheterh or not the twin of the criminal is also convicted does not depend on the biological type of the twin, we have a sample from a hypergeometric distribution

Indeed: there are 13 red (monozygote) balls and 17 black (dizygotic) balls. We randomly sample 12 balls (convicted), what is the probability that we will see 10 or more red balls in the same sample?

Let $X \sim Hyper(13,17,12) = (r,k,m)$

$$P(X \geq 10) = \sum_{k=0}^{12} \frac{\binom{13}{k}\binom{17}{12-k}}{\binom{30}{12}} \approx 0.000465$$

So- it seems very unlikely that there is no relation between the conviction of the twin and its type, but:

- Did we establish that a criminal mind is inherited?
    - Problem with this statement
    - Conviction vs truth "that face is up to no good??"
    - Sampling/ascertainment bias
    - Identical twins tighter connection?

Joint probability mass function:

The joint pmf of the RVs X and Y specifies their interaction:

$$P_{XY}(x,y) = P(X = x, Y = y) \;\; (x, y \in \mathbb{R})$$

Note: $\{X = x, Y = y\}; \{\omega \in \Omega | \, X(\omega) = x, Y(\Omega = y\}$

- If X and Y are discrete Random Variables, with
    - $\{x_i\} = X(\Omega); and \; \{y_i\} = Y(\Omega)$

Then: (keeping y fixed)

$$P_X(x) = P(X = x) = \sum_j P\big(X = x, Y = y_j\big) = \sum_j P_{XY}\big(x, y_j\big)$$

Similarly:

$$P_Y(y) = \sum_i P_{XY}(x_i, y)$$

P_X and P_y are referred to as the **marginal pmf's**

Example:

A fair coin is tossed 3 times

$$X = 1\{H\ on\ first\ toss\} = number\ of\ heads\ in\ first\ toss$$
$$Y = total\ number\ of\ heads$$

$$\Omega = \{HHH, HHT \dots\};\ |\Omega| = 8$$

Y	0	1	2	3	
X	1/8	2/8	1/8	0	1/2
0	0	1/8	2/8	1/8	1/2
1	0	1/8	2/8	1/8	
	1/8	3/8	3/8	1/8	

So, $P(X = 0) = \frac{1}{8}$

(note: called marginal, as it is in the margis)

NOTE:

$$X \sim Bernoull\left(\frac{1}{2}\right)$$
$$Y \sim Binomial\left(3, \frac{1}{2}\right)$$

Are X and Y independent RVs?

- The random variable X is indepdent of Y if "knowing the value of Y does not change the discibtion of X" (so NO- they are not independent)

Independence:

$$P(X = x_i, Y = y_j) = P(X = x_i, Y \neq y_j)$$

$$\rightarrow P(X = x_i, Y = y_j) = P(X = x_i)P(Y = y_j)\ \ (Definition)$$

$$P_{XY}(x, y) = P_X(x)P_Y(y)$$

The joint pmf factors into the product of the marginal

More generally, the random variables $X_1, \dots X_N$ are independent if:

$$P(X_1 = x_1, \dots X_n = x_n) = P_{X_1,\dots X_N}(x_1, \dots x_n) = \prod_i^n P_{X_i}(x_i) \quad (\forall x_i \in \mathbb{R})$$

Note: take caution that PAIWISE INDEPEDENCE DOES NOT IMPLY INDEPENDENCE

Poisson Distribution:

Recall:

$$X \sim Poisson(\lambda)$$

If

$$P_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

$$(k \in \mathbb{N}$$

- A 1 parameter faily, $\lambda > 0$
- Where does it come from?
  - Models the number of "events" that are registered in a certain time interval, eg: the number of
    - Particles emitted by a radioactive source in a hour
    - Incoming calls to a service centre between 1-2 pm
    - Light bulbs burnt in a year
    - Fatalities from horse kicks in the Prussian cavalry over 200 corp years (Boriewiez 1898)

20 corps x 10 years = 200 corp years

Number of deaths	Observed count	frequency	Poisson approximation
0	109	.545	.543
1	65	.325	.331
2	22	.110	.101
3	3	.015	.021
4	1	.005	.003

So, how did we come up with the Poisson distribution?

Assumptions of poission:

1. The distribution of the number of events of any time interval depends only on its length or duration. (eg; number of horse kicks only depends on that it is a day, not on the particular day)

2. The number of events recorded in two disjoint time intervals are independent of one another (numer of horse kicks is independent from today and yesterday)
3. No two events are recorded at exactly the same time point (you can't have 2 horsekicks exactly at the same time, must be slightly different in time)

So:

Let $X_{t,s}$ denote the number of events in the time intercal $(t, s]$

- Denote by: $X_t = X_{0,t}$

Our goal is to find the disctibution of X:

- Let $f(t) = P(X_t = 0)$, then

$$\begin{aligned}
&f(t+s) = P(X_{t+s} = 0) \\
&= P\big(X_t = 0, X_{t,s+t} = 0\big)\ (disjoint\ time\ intervals, and\ using\ property\ 2) \\
&= P(X_t = 0)P\big(X_{t,s+t} = 0\big)\ \ (property\ 1) \\
&= P(X_t = 0)P(X_s = 0) = f(t) = f(s) \\
&\rightarrow f(t+s) = f(t)f(s)\ \forall t, s > 0
\end{aligned}$$

One type of solution of this is:

$$f(t) = e^{\alpha t}\ \ (\alpha \in \mathbb{R})$$

(note: other solutions exists, but are a mess, in that they are unbounded, but $f(t) \in [0,1]$, so can't exist for probabilities).

For the same reason, $\alpha < 0$ (to remain bounded)

So:

$$\begin{aligned}
&\alpha = -\lambda\ \ (for\ \lambda > 0) \\
&\rightarrow P(X_t = 0) = e^{-\lambda t}\ \ (for\ t = 1) \\
&P_{X_1}(0) = P(X_1 = 0) = e^{-(\lambda)}
\end{aligned}$$

Let: $Y_n =$ the number of intervals in $\left(\frac{k-1}{n}, \frac{k}{n}\right]$ in $(0,1]$ where an event occurred is:

$$\begin{aligned}
&= \sum_{k=1}^{n} 1_{\left\{X_{\frac{k-1}{n},\frac{k}{n}} \geq 1\right\}}\ \ \big(n\ \ iid\ Bernoulli\ (p_n)\big)\ \ (as\ independent\ of\ time\ interval, and\ disjoint) \\
&\therefore p_n = P\left(X_{\frac{k-1}{n},\frac{k}{n}} \geq 1\right) \\
&= P\left(X_{\frac{1}{n}} \geq 1\right)\ \ \ \ (a\ least\ 1) \\
&= 1 - P\left(X_{\frac{1}{n}} = 0\right)\ \ (law\ of\ total\ probabilities)
\end{aligned}$$

$= 1 - e^{-\frac{\lambda}{n}}$

$$\to \frac{1}{n} \sim Binomial\left(n, p_n = 1 - e^{-\frac{\lambda}{n}}\right)$$

Note: $\frac{1}{n} \leq X_1$ and $Y_n < X_1$, if two events occur in the same interval

However:

$$\lim_{n\to\infty} Y_n = X_1$$

(as X_1 accounts ALL events, but Y_n counts only the events in a certain time interval)

Because no two events can occur at the same time. (property 3)

The expected value of a $binomial(n, p)$ RV is np. For $\frac{1}{n}$, this is $np_n = n\left(1 - e^{\frac{\lambda}{n}}\right)$, what is the $\lim_{n\to\infty} np_n$?

$$= n\left(1 - \left(1 - \frac{\lambda}{n} - R_1\left(-\frac{\lambda}{n}\right)\right)\right) \ (using\ the\ Taylor\ expansion)$$

$$= \lambda + \frac{R_1\left(-\frac{\lambda}{n}\right)}{-\frac{\lambda}{n}} \lambda \to \lambda$$

Therefore, the limit of the expected value for the binomial is λ

Claim: if $Y_n \sim Binom(n, p_n)$ such that $np_n \to \lambda\ (as\ n \to \infty)$, then for any fixed $k \in \mathbb{Z}^+$:

$$P(Y_n = k) \to_{n\to\infty} e^{-\lambda} \frac{\lambda^k}{k!}$$

(binomial converges to the poisson pmf)

Corollary:

$$X = X_1 \sim Poisson(\lambda)\ for\ large\ n$$
$$\lim_{n\to\infty} P(Y_n = k) \to P(X_{poisson})$$

Proof of claim:

$$P(Y_n = k) = \binom{n}{k} p_n^k (1 - p_n)^{n-k}$$
$$= \frac{n(n-1) \ldots (n-k+1)}{k!} p_n^k$$

As $n \gg k$:

$$\rightarrow \frac{(np_n)^k}{k!}\left(\frac{n}{n}\frac{n-1}{n}\dots\frac{n-k+1}{n}\right)$$
$$as\ n \rightarrow \infty = \frac{\lambda^k}{k!}\ (using\ assumption)$$

Also:

$$(1-p_n)^{n-k} = (1-p_n)^n(1-p_n)^k$$
$$\rightarrow (1-p_n)^n\ \ (as\ p_n \rightarrow 0\ as\ n \rightarrow \infty)$$
$$\therefore \lim_{n\to\infty}(1-p_n)^n = e^{n\ln(1-p_n)}$$
$$Taylor\ expansion: \ln(1-x) = -x + R_1(x),\ \ where\ \frac{R(x)}{x} \rightarrow 0$$
$$\therefore limit\ is:$$
$$nln(1-p_n) = n[-p_n + R_1(p_n)]$$
$$= -np_n + \frac{R_1(p_n)}{p_n}(np_n)$$
$$\therefore limit \rightarrow\ -\lambda$$
$$\therefore \lim_{n\to\infty}((1-p+n)^n) = e^{-\lambda}$$

Example of poisson approximarion to binomial distribution:

$$n = 100; p = 0.01; \lambda = np1$$
$$P(Y_n = 3) = 0.0610\ \ Y_n \sim Binom(n, p)$$
$$P(X_{Poisson} = 3) \approx 0.0613\ \ \ (X \sim Poisson(\lambda))$$

(so pretty good approximation)

Proof of corollary:

$$\lim_{n\to\infty} P(Y_n = k)$$

Going back to our horse deaths:

What is missing?

λ is missing, how did he come up with his lambda?

Computing the mean of the empirical data

Example: faulty monitor

The radioative source gives off particles emitted according to $Poisson(\lambda)$

How do we see if the monitor is faulty, and if it it picking up all the radiation?

Let X be the number of particle REGISTERED in an hour

What is the distribution of X?

Let $N =$ the number of particles EMITTED in the same time (note: it is unobserved)

For $k \in \mathbb{N}$;

$$P(X = k) = \sum_{k=0}^{n} P(X = n|N = n)P(N = n) \quad (using\ law\ of\ total\ probability)$$

$$note: P(X = n|N = n) = 0\ for\ all\ n \leq k\ \ (it\ can't\ see\ particles\ which\ aren't\ there)$$

First is is a binomial distribution; $Binom(n, p)$; second is poisson (given)

$$so\ P(X = k) = \sum_{k=0}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} e^{-\lambda} \frac{\lambda^n}{n!}$$

$$= e^{-\lambda} \frac{(\lambda p)^k}{k!} \sum_{n=0}^{\infty} \frac{\lambda^{n-k}(1-p)^{n-k}}{(n-k)!} \quad (doing\ some\ algebra)$$

$$letting\ m = n - k$$

$$= e^{-\lambda} \frac{(\lambda p)^k}{k!} \sum_{n=0}^{\infty} \frac{\left(\lambda(1-p)\right)^m}{(m)!}$$

$$= e^{-\lambda} \frac{(\lambda p)^k}{k!} e^{\lambda(1-p)} \quad (taylor\ series)$$

$$= e^{-\lambda p} \frac{(\lambda p)^k}{k!}$$

SO:

$$X \sim Poisson(\lambda p)$$

This makes sense, that the three counting assumptions for the poisson distribution still hold

Distibution of a sum of random variables: (arithmetricies of RV)

If X and Y are jointly distributed $\mathbb{Z}^+$ valued RV's; then $Z = X + Y$ is also a Z^+ valued RV, and for $n \in \mathbb{Z}^+$:

Pmf of Z:

$$P_Z(n) = P(Z = n)$$

$$= \sum_{k=0}^{n} P(X = k, Y = n - k)$$

$$= \sum_{k=0}^{n} p_{XY}(k, n - k)$$

If X and Y are independent; then:

$$p_z(n) = \sum_{k=0}^{n} p_X(k) p_Y(n - k)$$

Convolution:

p_Z is the **convolution** of p_X and p_Y

$$p_Z = p_X * p_Y$$

Example:

1. $X \sim Binomial(m, p)$ is independent of $Y \sim Binom(n-m, p)$

$$Z = X + Y$$

It is clear that $Z \sim Binom(n, p)$

Formally:

$$l = \{0, \dots n\}$$

$$P(Z = l) = \sum_{k=0}^{l} \binom{m}{k} p^k (1-p)^{m-k} \binom{n-m}{l-k} p^{l-k} (1-p)^{n-m-(l-k)}$$

$$= p^l (1-p)^{n-l} \sum_{k=0}^{l} \binom{m}{k} \binom{n-m}{l-k}$$

$$= p^l (1-p)^{n-l} \binom{n}{l}$$

Proof that:

$$\sum_{k=0}^{l} \binom{m}{k} \binom{n-m}{l-k} = \binom{n}{l}$$

Combinatorics:

$\binom{n}{l}$ is the number of committees of l students from a class of n

Suppose the class has m boys and $n-m$ girls; then the same number can be counted as

$$\sum_{k=0}^{l} \{number\ of\ committees\ with\ k\ boys\} = \sum_{k=0}^{l} \binom{m}{k} \binom{n-m}{l-k}$$

Algebraic proof:

$$(1+x)^n = (1+x)^n (1+x)^{n-m}$$

consider the coefficient of x^l on both sides

$$LHS = \binom{n}{l};\ \ RHS = \sum_{k=0}^{l} \binom{m}{k} \binom{n-m}{l-k}$$

Example 2:

$$X \sim POisson(\lambda);\ \ Y \sim POisson\ (\gamma);\ \ what\ is\ P(Z = X + Y)?$$

$$P(Z=n)=\sum_{k=0}^{n} p_X(k)p_Y(n-k)=\sum_{k=0}^{n}\frac{e^{-\lambda}\lambda^k}{k!}\frac{e^{-\gamma}\gamma^k}{(n-k)!}$$

Example 3

$$X\sim\ B(n,p); Y\sim B(n-m,p)$$

We already saw $X+Y\sim B(n,p)$, ut now condider the CONDITIONAL distribution of X, given $X+Y$

For $k=0,\ldots n$

$X\sim Poisson(\lambda)$ and $Y\sim P(\gamma)$; we say $X+Y\sim Po(\lambda+\gamma)$

The conditional distribution of X is

$$P(X=k|X+Y=n)=\frac{P(X=k,Y=n-k)}{P(X+Y=n)}=\frac{\frac{e^{-\lambda}\lambda^k}{k!}\frac{e^{-\gamma}\gamma^k}{(n-k)!}}{\frac{e^{-\lambda-\gamma}(\lambda+\gamma)^n}{n!}}$$

$$\rightarrow X\sim Binom\left(n,\frac{\lambda}{\lambda+\gamma}\right)$$

Imagine the sources emitting particles at rates λ and γ

Expectation:

Definition

Def: the expected value, or expectation, of a discrete non-negative RV X is the weighted average with range $X(\Omega)=\{x_i\}$ is given by:

$$E(X)=\sum_i x_i p_i(x_i)$$

Well definied for positive and negative separately; not well definied for if it is positive and negative

Examples of particular kinds

Bernoulli:

$$X\sim Bernoulli(p)$$
$$E(X)=0(1-p)+1(p)=p$$

Poisson

$$X\sim POisson(\lambda)$$

$$E(X) = \sum_k \frac{ke^{-\lambda}\lambda^k}{k!}$$
$$= \lambda e^{-\lambda} e^{\lambda}$$
$$= \lambda$$

Binomial

$$= \sum_k k\binom{n}{k} p^k (n-p)^{n-k}$$
$$= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\,[(n-1)-(k-1)]} p^{k-1}(1-p)^{n-1-(k-1)}$$
$$= np \sum_{m=1}^{n-1} \binom{n-1}{m} p^m (1-p)^{n-m}$$
$$= np\big(p + (1-p)\big)$$
$$= np$$

Geometric:

$$E(X) = \sum_{k=1}^{\infty} k(1-p)^{k-1} p$$

Consider the power series:

$$f(q) = \sum_{k=0}^{m} q^k = \frac{1}{1-q} \quad for\ |q| < 1$$

A power series can be differentiaed term by term within its radius of convergence

$$\rightarrow f'(q) = \sum_{k=1}^{m} kq^{k-1} \quad \forall q \in (-1,1)$$
$$but\ f'(q) = \frac{d}{dq}\left(\frac{1}{1-q}\right) = \frac{1}{(1-q)^2}$$
$$\rightarrow \frac{1}{(1-q)^2} = \sum_{k=1}^{m} kq^{k-1}$$
$$\therefore E(X) = p \sum_{k=1}^{\infty} \frac{kq^{n-1}p}{(1-q)^2}$$
$$= \frac{1}{p}$$

Negative binomial

$$E(X) = \frac{r}{p}$$