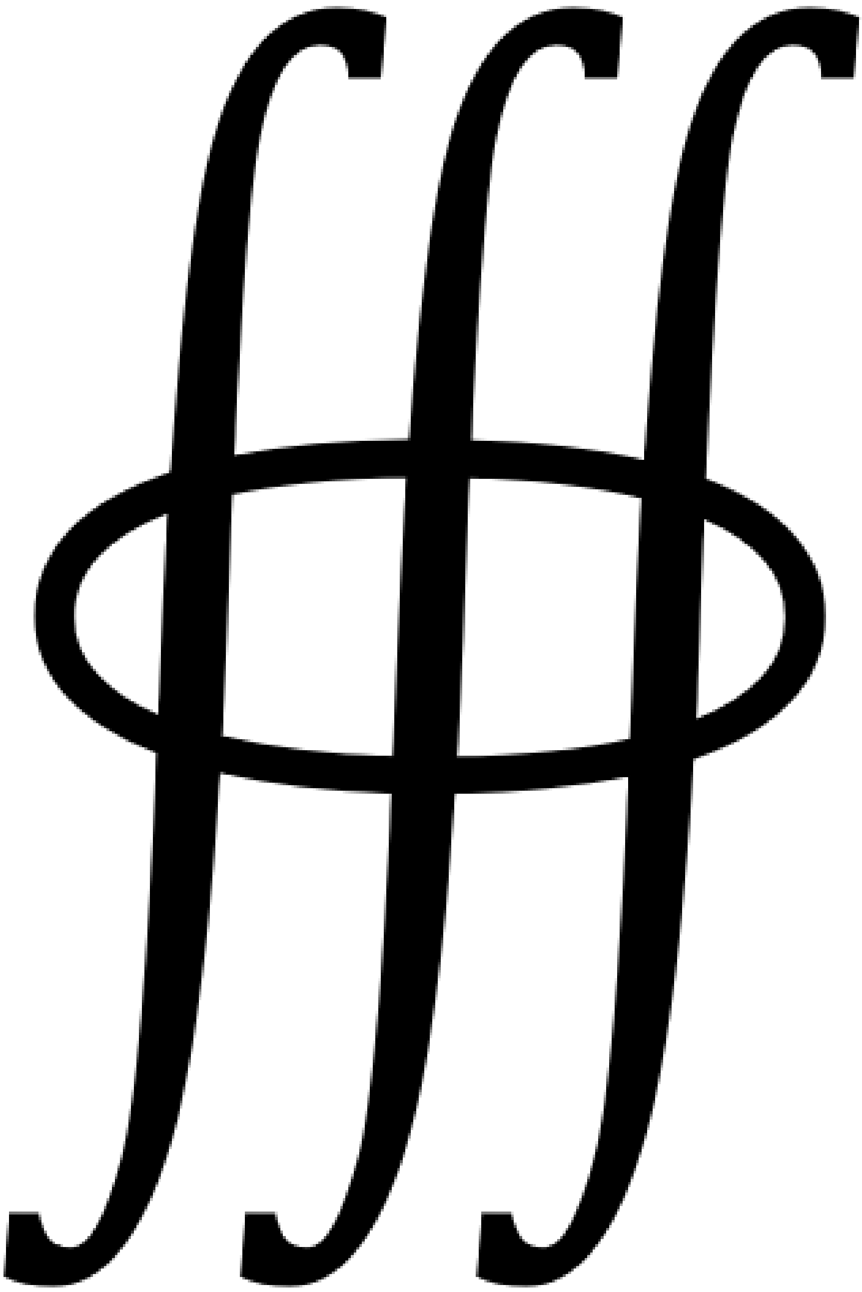


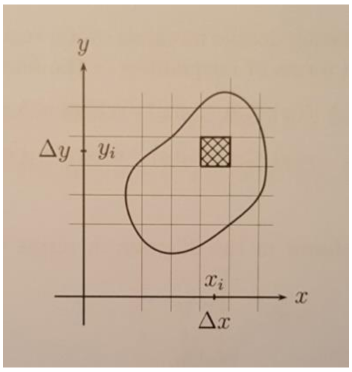
# VECTOR CALCULUS

Julian.O 2016

Vector Calculus Lecture 3:

Double Integrals
Green's Theorem
Divergence of a Vector Field

Double Integrals:
Double integrals are used to integrate two-variable functions $f(x,y)$ over a region $R$ in the $xy$-plane. The theory behind double integrals involves splitting the region up into small rectangles with dimensions $\Delta x$ and $\Delta y$.



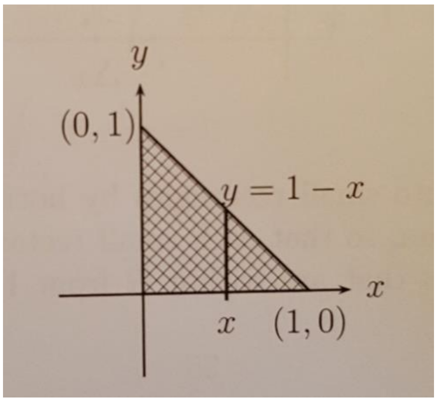

Integrating a function $f(x,y)$ over this region is then a matter of summing the value of the function multiplied by the dimensions of each rectangle for each rectangle. Taking the limit as $\Delta x$ and $\Delta y$ approach zero produces the double integral with respect to $dA = dxdy$.

$$\iint_R f(x,y)\, dA = \iint_R f(x,y)\, dxdy$$

Double integrals are evaluated by performing integration twice, once with respect to $x$ and once with respect to $y$. In order to find the limits for these integrals, we need to express $R$ using inequalities in terms of $x$ and $y$. This often takes one of the following forms. We are often required to express one of these variables in terms of the other.

$a \leq x \leq b, f(x) \leq y \leq g(x)$ or $c \leq y \leq d, p(y) \leq x \leq q(y)$

Consider the following examples.

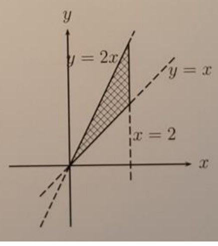



$$\{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1 - x\}.$$

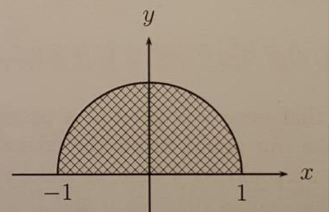
$$\{(x,y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1,\ 0 \leq x \leq 1 - y\}.$$

$R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2,\ x \leq y \leq 2x\}$.

The region is enclosed by the lines $x = 0,\ x = 2,\ y = x,\ y = 2x$.




The shaded region is half a unit circle, $\{(x, y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1,\ 0 \leq y \leq \sqrt{1 - x^2}\}$.




When we evaluate double integrals, we have to evaluate with respect to the variable that has been expressed in terms of the other variable first. For example, if we have expressed the region using the inequalities $0 \leq x \leq 1$ and $0 \leq y \leq 1 - x$ then we have to evaluate the definite integral with respect to $y$ first. This will eliminate $y$ so we can evaluate with respect to $x$ to produce a number. Consider the following example.

$$\begin{aligned}\iint_R (x^2 + y^2)\,dx\,dy &= \int_0^1 \left\{ \int_0^{1-x} (x^2 + y^2)\,dy \right\} dx \\ &= \int_0^1 \left\{ \left[ x^2 y + \frac{y^3}{3} \right]_0^{1-x} \right\} dx \\ &= \int_0^1 \left[ x^2(1 - x) + \frac{(1 - x)^3}{3} \right] dx = \frac{1}{6}.\end{aligned}$$

Double integrals allow us to calculate three different physical quantities in 3D space: area, volume, and mass. If the region $R$ is the region defined by $a \leq x \leq b$ and $f(x) \leq y \leq g(x)$, then the area of $R$ is given by the following double integral.

$$Area = \iint_R dA = \int_a^b [g(x) - f(x)]\,dx$$

If a function $f(x, y)$ is greater than 0 for all values in some $xy$-region $R$ then $z = f(x, y)$ represents the surface sitting over $R$ in the $xy$-plane. The volume of the solid formed between the region $R$ on the $xy$-plane and the function $z = f(x, y)$ is then given by the following double integral.

$$Volume = \iint_R f(x, y)\,dxdy$$

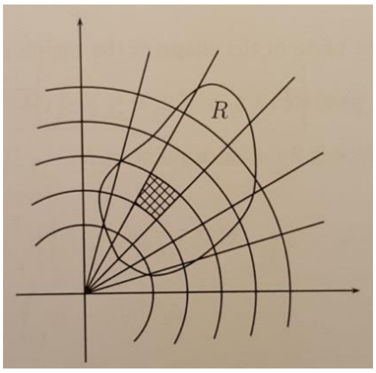
Suppose that a infinitely thin material in the shape of some region $R$ in the $xy$-plane has a density characterised by the function $f(x, y)$. The mass of the material is then given by the following double integral.

$$Mass = \iint_R f(x, y)\,dxdy$$

Double integrals that are defined over a disc-like region can be more easily evaluated using polar coordinates rather than Cartesian coordinates. This means we express everything in terms of the radius $r$ and the angle from the positive $x$-axis $\theta$. We use the following equations to convert from Cartesians coordinates to polar coordinates.

$x = r\cos\theta$
$y = r\sin\theta$

Since we are now integrating a function over a region in polar coordinates, the region is split up using origin-centred circles and rays emanating from the origin.




As done previously for Cartesian coordinates, taking the limits as $\Delta r$ and $\Delta\theta$ approach zero produces the double integral with respect to $dA = rdrd\theta$.

$$\iint_R f(x,y)\,dA = \iint_R f(r\cos\theta\,, r\sin\theta)\,rdrd\theta$$

Green's Theorem:
Green's theorem states that a double integral over a plane region is equal to a line integral over the boundary of the region. We can use this formula to more easily evaluate difficult line integrals using double integrals.

$$\oint_C \boldsymbol{F}.dr = \oint_C F_1dx + F_2dy = \iint_R \left(\frac{\delta F_2}{\delta x} - \frac{\delta F_1}{\delta y}\right) dxdy$$

Combining Green's theorem and the double integral formula for area, we can derive an additional formula for calculating the area of a region $R$ using a line integral over the closed curve $C$ bounding the region.

$$Area\ of\ R = \frac{1}{2}\oint_C -ydx + xdy$$

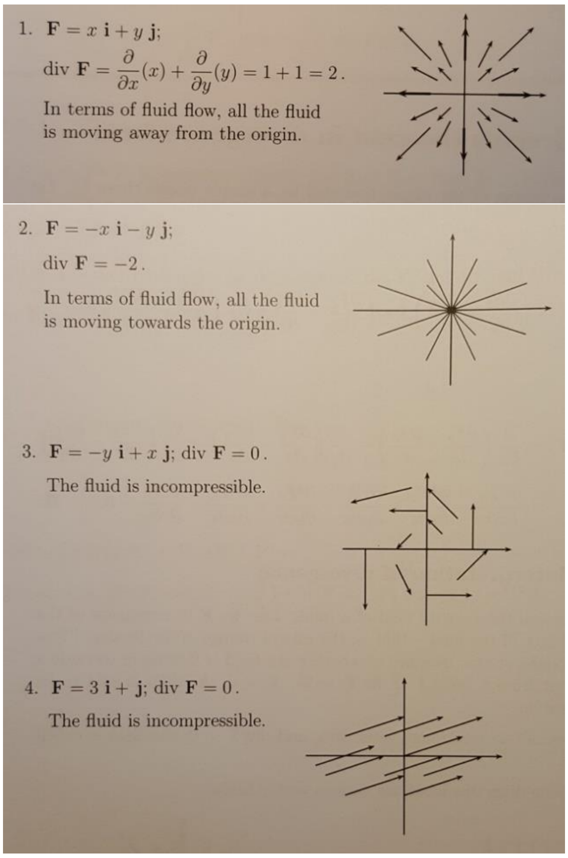
The formula for Green's theorem can be rewritten in vector form using the curl of an arbitrary vector field $\boldsymbol{F}(x,y) = F_1(x,y)i + F_2(x,y)j$.

$$\oint_C \boldsymbol{F}.dr = \oint_C F_1dx + F_2dy = \iint_R (\nabla \times \boldsymbol{F}).k\ dxdy$$

Divergence of a Vector Field:
The divergence of a vector field $\boldsymbol{F}$ is the scalar dot product of the vector field and the del operator $\nabla$. The divergence of a vector field is a measure of the extent to which the vectors at a point are travelling inwards, outwards, or neither.

$$div\,\boldsymbol{F} = \nabla.\boldsymbol{F} = \frac{\delta F_1}{\delta x} + \frac{\delta F_2}{\delta y} + \frac{\delta F_3}{\delta z}$$

Consider the following examples of vector fields and how the divergence is used to describe whether the origin is a source, sink, or neither.



1. $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$;

   $\text{div}\,\mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 1 + 1 = 2$.

   In terms of fluid flow, all the fluid is moving away from the origin.

2. $\mathbf{F} = -x\,\mathbf{i} - y\,\mathbf{j}$;

   $\text{div}\,\mathbf{F} = -2$.

   In terms of fluid flow, all the fluid is moving towards the origin.

3. $\mathbf{F} = -y\,\mathbf{i} + x\,\mathbf{j}$; $\text{div}\,\mathbf{F} = 0$.

   The fluid is incompressible.

4. $\mathbf{F} = 3\,\mathbf{i} + \mathbf{j}$; $\text{div}\,\mathbf{F} = 0$.

   The fluid is incompressible.

If the divergence of a vector field is 0, then the vector field is said to be divergenceless or solenoidal. Note that the divergence of the curl of a vector field is always 0.

$$div(curl\,\boldsymbol{F}) = 0$$

Finding the double integral of the divergence of an arbitrary vector field $\boldsymbol{F} = F_1 i + F_2 j$ and using Green's theorem produces the following equation involving the normal to the curve $C$. This is known as the flux of the vector field $\boldsymbol{F}$ over a curve $C$.

$$Flux\ of\ \boldsymbol{F}\ over\ C = \oint_C \boldsymbol{F}.\boldsymbol{n}\,ds = \iint_R div\,\boldsymbol{F}\,dxdy$$