MATH2070 optimisation and financial maths lecture notes

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OPTIMISING DIFFERENTIALE FUNCTIONS

CHAPTER 1: OPTIMISING DIFFERENTIABLE FUNCTIONS

Examples: physics, chemical reactions, scheduling, manufacturing.

Eg: Minimising Surface area of can

Start by modelling the can as a cylinder:

$$V = \pi r^2 h = 375mL$$
$$S = 2\pi r + 2\pi r^2$$
$$h = \frac{V}{\pi r^2}$$
$$\Longrightarrow S(\mathrm{r}) = \frac{2\pi}{r} + 2\pi r^2$$
$$\therefore differentiate\ ect$$

$$\therefore r = \left(\frac{V_0}{r\pi}\right)^{\frac{1}{3}}$$

But: this gives $r \approx 3.91; h = 7.82$. So why is this different to the ACTUAL size of a can?

Modelling in incorrect (eg- S has no width, indentation at bottom ect)

$$industrial\ paramenters\colon\ d_{side} = 0.0104cm$$
$$d_{top} = 0.0236cm$$
$$d_{bottom} = 0.0203cm$$

$$\therefore \bar{S}(h,r)\ (not\ an\ area) = 2\pi rhd_{side} + \pi r^2 d_{bottom} + \pi r^2 d_{top}$$
$$\therefore \bar{S}(r) = 2\pi r\left(\frac{V}{\pi r^2}\right)d_{side} + \pi r^2\left(d_{bottom} + d_{top}\right) = \frac{2V}{r}d_{side} + \pi r^2(d_{bottom} + d_{side})$$
$$\frac{d\bar{S}}{dr} = -\frac{2V}{r^2}d_{side} + 2\pi r(d_{bottom} + d_{side}) = 0$$

Mathematical optimisation:

Given an **objective function**, $f\colon \mathbb{R}^n \to \mathbb{R}\quad (\text{scalar function})$

And a **feasible region**: Ψ

And **optimisation problem** is the problem of finding an $x^* \in \mathbb{R}^n$ that solves:

$$\min_{x\in\mathbb{R}^n} f(x)\ \ |x \in \Psi\ \ or\ \ \max_{x\in\mathbb{R}^n} f(x)\ \ |x \in \Psi$$

Optimisation of differentiable functions of one variable

Some scenarios:

$$f(x)\ is\ constant\ for\ x \in [a,b]$$
$$\Longrightarrow all\ x \in [a,b]\ optimised$$

$$f(x)\ is\ linear\ for\ x \in [a,b]$$
$$\therefore optimised\ points\ on\ boundary$$

$$f(x)\ has\ unique\ global\ extremity\ in\ interior:$$

$$f(x)\ has\ multiple\ local\ maximum\ or\ minimum$$
$$must\ use\ computers\ and\ find\ an\ algorithm\ to\ solve\ it$$

Global minimum and maximum

Definition: a point x^* is a **global minimum** if $f(x^*) \leq f(x)\ \forall x \in \Psi$

Definition: a point x^* is a **local minimum** if there is a neighbourhood N of $x^* | f(x^*) \leq f(x) \forall x \in N$

Identifying local extremities of $f(x)$

1. First derivative test $f'(x^*) = 0$ (could be min, max or inflexion), only necessary condition for the existence of optimal
2. Sufficient condition can be established using higher order derivatives:
   - $f'(x^*) = 0;\ f''(x^*) < 0$: local maximum
     But: eg this would not mind max of $-x^4$
   - If $f'(x^*) = f''(x^*) = \cdots = f^{2m-1}(x^*) = 0$ and
     $$f^{2m}(x^*) < (>) 0, x^*\ is\ maximum (minimum)$$
   - If $f^1(x^*) = \cdots = f^{2m}(x^*) = 0$, and $f^{2m-1}(x^*) \neq 0$, then x^* is a point of inflection

Finding global extremity:

Now we can test for global extremity:

$$\min\{f(a), f(b), f(x_1^*), f(x_2^*) \ldots, f(x_k^*)\}$$

(or max)

Revision of solving linear equations:

Eg:

$$Ax = b$$
$$x = A^{-1}b$$

Pivot operation algorithm:

Eg solve:

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix}$$

1. Decide on a **pivot element**, $a_{ij} \neq 0$
2. Divide row i by $a_{ij} \neq 0$ (in our lecture, a_{ij} are large enough to not amplify errors (that it may in a computer))
3. Transform all other rows of a_{kj} $(k \neq i)$ by adding suitable multiples of row i

Eg: in tableaux form

x_1	x_2	x_3	b
1	-1	1	-2
2	1	-1	5
-1	2	3	0

x_1	x_2	x_3	b
1	-1	1	-2
0	3	-3	9
0	1	4	-2

x_1	x_2	x_3	b
1	0	5	-4
0	0	-15	15
0	1	4	-2

$$x_3 = -1; x_2 = -2 - 4(-1) = 2; x_1 = -4 - 5(-1) = 1$$

$$\therefore \boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Transformation of linear functions with Gaussian Jordan elimination

Basic/nonbasic variables

If we were given:

$$\begin{pmatrix} 1 & -1 & 1 & -1 & 0 \\ 2 & 1 & -1 & 0 & 1 \\ -1 & 2 & 3 & 1 & 2 \end{pmatrix}\boldsymbol{x} = \begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix}$$

As this system has infinite solutions, we can simplify a solution $Z = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5x_5 + c_0$ into the form:

$$Z = Ax_4 + Bx_5 + C$$

(as x_1, x_2 *and* x_3 can be expressed in terms of x_4 and x_4

In this case: the variables which have a unique solution are known as **non-basic**, whereas the one's which do not (x_4, x_5) are called **basic**

Eg: the system above simplifies to:

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{8}{15} & \frac{2}{3} \\ 0 & 0 & 1 & -\frac{2}{15} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

So: $x_1 = 1 + \frac{1}{3}x_4 - \frac{1}{3}x_5$; $x_2 = 2 - \frac{8}{15}x_4 - \frac{2}{3}x_5$; $x_3 = -1 + \frac{2}{5}x_4 - \frac{1}{3}x_5$

LINEAR PROGRAMMING

CHAPTER 2: LINEAR PROGRAMMING

- The term **programming** means planning/logistics (not computing)
  - Used for : allocating limited resources among competing activities in optimal way
  - Selecting the level of certain activities that compete for limited resources to optimise some objective function
    - Eg:
      - Resource allocation
      - Portfolio selection
      - Transportation
      - Agriculture
      - Manufacturing

Standard LP Problem:

I. Maximise $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

II. Subject to:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & \dots & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} \leq \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix}$$

III. With: $x_1, x_2, \dots x_n \geq 0$

OR:

Maximise $Z = \boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x}$

Subject to $A\boldsymbol{x} \leq \boldsymbol{b}$

And $\boldsymbol{x} \geq \boldsymbol{0}$

I. Z is the **objective function.** It is a linear function of the **decision variables** $(x_1, x_2, \dots, x_n)$. The constants $(c_1, c_2, \dots, c_n)$ are the **cost coefficients**. The increase in Z for unit increase in x_k is c_k.

II. This part states the **linear constraints** of the problem. The coefficient matrix is the **constraint matrix.** In standard LP problems, all elements of the **resource vector** $(b_1, b_2, \dots, b_n)$ are assumed to be non-negative.

III. The final part of the LP problem is the **positivity condition**: of the decision variables $(x_1, x_2, \dots, x_n)$

Any $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ that satisfy II and III are **feasible solutions**, and lie in a closed region in the decision space, called the **feasible region:** Ψ**.** The decision space is always non-empty as $(0,0,\dots 0)$ is always feasible.

Any $\boldsymbol{x}$ not in the feasible region is **infeasible**.

A feasible solution of $\boldsymbol{x}$ which maximimes the objective function Z is the **optimal solution.** Denoted $\boldsymbol{x}^*$.

As the objective function is Linear (in standard LP problems), the maximum and minimum of Z must lie on the boundary of the feasible region.

Example of LP problem:

	Resources (P_4) needed percent of product		
	Product		
"competing" sites	white	blue	Amount of resources available
RV_1	1	0	4
RV_2	3	2	18
RV_3	0	2	12
Objective function Z	3	5	

$$\therefore LP\ problem\ is:$$

If x_1 is the number of white units

x_2 is the number of blue units:

$$\therefore Maximise: Z = 3x_2 + 5x_2$$

Subject to:

$$x_1 \leq 4$$
$$3x_1 + 2x_2 \leq 18$$
$$2x_2 \leq 12$$

And $x_1, x_2 \geq 0$