DISCRETE MATHS AND GRAPH THEORY: MATH2069 summary

Contents

Chapter 1: counting principles

1. Study of non-continuous maths
2. (eg, sequences, combinatorics, induction ect)

Eg: continuous function $y = x^2$; has analogous sequences $a_n = n^2$

$$\mathbb{N} = \{0,1,2 \dots\}$$

For finite set $X = \{x_1, x_2 \dots x_n\}, |X| = n$

The size (or cardinality) of X

Theorem: bijecetion principle

If 2 sets X and Y are a bijection; $|X| = |Y|$

3. Bijection is 1-1 correspondence, and is a way of associating to each element of X a corresponding element of Y

Injective:

1-1: $f: X \to Y$ if $f(x_1) \neq f(x_2)\ \ for\ x_1 = x_2$

For each $y \in Y$, there is at most 1 $x \in X | f(x) = y$

Surjective:

'onto': $f: X \to Y$, if each value of Y is one of the values taken by the function (eg, if the function maps to all $\mathbb{N}$, then $Y = \mathbb{N}$

Bijective= surjective+injective

Sum principle

If $X = A \cup B\ and\ A \cap B = \emptyset$

$$|X| = |A| + |B|$$

Difference principle

For any $A \subset X$

$$|X \backslash A| = |X| - |A|$$

Preimage: $f^{-1}(y) \coloneqq \{x \in X | f(x) = y\}$

$$|X| = \sum_{y \in Y} |f^{-1}(y)|$$

Rounding notation:

$$\lceil x \rceil \coloneqq x\ rounded\ up\ to\ nearest\ integer$$
$$\lfloor x \rfloor \coloneqq x\ rounded\ down\ to\ nearest\ integer$$

$$\lfloor x \rceil = x\ rounded\ (up\ or\ down) to\ nearest\ integer$$

$$x - 1 < \lfloor x \rfloor \leq x; \quad x \leq \lceil x \rceil < x + 1$$

Pigeonhole principle:

$$f: X \to Y, there\ must\ be\ a\ y \in Y |$$
$$|f^{-1}(y)| \geq \left\lceil \frac{|X|}{|Y|} \right\rceil$$

Cartesian product:

$$X \times Y \coloneqq \{(x, y) | x \in X, y \in Y\}$$

Product principle:

$$|A| \times |B| \times |C| \times \dots = |A \times B \times C \dots|$$
$$|X^n| = |X|^n$$

Overcounting principle:

If $f: X \to Y, and\ |f^{-1}(y)| = m$

$$then\ |Y| = \frac{|X|}{m}$$

Number of subsets of a set:

If $Y \subset X = \{1,2, \dots n\}$

$$\therefore number\ of\ subses\ Y = 2^n$$

Number of ordered selections $(y_1 \dots y_k)$

Injective functions

Is analogous to umber of injective functions

$$number\ of\ injective\ function's from\ X \to Y\ is$$

$$= \boldsymbol{n_k} = \frac{n!}{(n-k)!} \quad ((-m)! = \infty) \qquad (n)(n-1)(n-2) + (n-k+1)$$

Number of unordered selections:

$$= \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Binomial theorem:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Proof: $(a+b)^n = (a+b)(a+b)\dots(a+b) \;\; (n\ times)$

$timesing\ everything\ together\ with\ the\ distributive\ law\ will\ get\ multiples\ of\ a^{n-k}b^{k}\ '$

Eg: $aababbb \dots ba\ ect$

$$number\ of\ ways\ to\ choose\ where\ b\ sits\ from\ n\ places\ is\ \binom{n}{k}$$

Properties:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\sum_{n=0}^{n} \binom{n}{k} = 2^n$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \begin{cases} 0 & n \geq 1 \\ 1 & n = 0 \end{cases}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = (\binom{2n}{n}$$

Multinomial coefficients:

Let X be a set with $|X| = n$, and $n_1, n_2 \dots n_m \in \mathbb{N} | \sum n_i = n$

$$let\ \binom{n}{n_1, n_2, n_3 \dots n_m} = the\ number\ of\ ways\ of\ choosing\ subsets\ A_1 \dots A_m$$

From $X; |A_i| = n_i; X = A_1 \cup A_2 \dots \cup A_m\ (disjoint\ union)$

$$\binom{n}{n_1, n_2, n_3 \dots n_m} = \frac{n!}{n_1!\, n_2!\, n_3! \dots n_m!}$$

Proof: if X lies on a line $X = 1,2 \dots n$

$number\ of\ permuations\ of\ X = n!$

$if\ we\ were\ to\ choose\ the\ first\ n_1\ to\ be\ in\ A_1, the\ net\ n_2 to\ be\ in\ A_2 ect$

$then\ the\ number\ of\ ways\ to\ rearagne\ X\ ino\ subsets\ A_i\ (where\ the\ subset\ A_i\ then\ stays\ the\ same)$

$$= \frac{n!}{n_1!\, n_2! \dots n_m!}$$

Eg: if there were 12 balls, and we wanted to divide them into 4 jars, of 2,3,5 and 2 balls in each respectively. Number of ways this can be done is $\binom{12}{2,3,5,2}$

Multinomial theorem:

$$(a_1 + a_2 + \cdots + a_m)^n = \sum_{n_1 + n_2 + \cdots + n_m = n} \binom{n}{n_1, n_2, n_3 \dots n_m} a_1^{n_1} a_2^{n_2} \dots a_n^{n_m}$$

Proof analogous to binomial theorem:

Multinomial and binomial link:

$$\binom{n}{n_1, n_2, n_3 \dots n_m} = \binom{n}{n_m} \binom{n - n_m}{n_{m-1}} \dots \binom{n_1}{n_1} = \prod_{i=1}^{m} \binom{\sum_{j=1}^{i} k_j}{k_i}$$

Unordered selections with repetition

The number of unordered selections with repetition is:

$$\binom{n + k - 1}{k}$$

Proof: having chosen elements of each type in a row: eg type 1, type 2.. type n (where $t_1 + t_2 + \cdots + t_n = k$)

We can separate these groups of each type with $n-1$ lines

Total objects= $n+k-1$

Number of ways to choose placemet of each object is $\binom{n+k-1}{k}$

Eg: the number of ways to give 3 identical boxes to 5 people, and each person can get multiple boxes:

$$= \binom{3+5-1}{3} = \binom{7}{3}$$

Inclusion/ exlusion principle:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Ect:

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum|A_i| - \sum|A_i \cap A_j| + \sum|A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1}\sum|A_1 \cap A_2 \cap \ldots \cap A_n|$$

$$\therefore |A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{k=1}^{n}\left[(-1)^{k-1} \sum_{i_1,i_2,i_3\ldots i_n=1}^{n} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_n}|\right]$$

Proof: suppose I is the subset $I = \{1,2,3 \ldots n\}, of\ each\ x \in A_i$

Then $A_{i_1} \cap A_{i_2} \cap \ldots A_{i_k}$ contains $x\ iff\ \ (i_1, i_2, i_3 \ldots i_k \in I)$. So, for each k, x is counted $\binom{|I|}{k}$

So the total number on the RHS = $\sum_{k=1}^{|I|}\binom{|I|}{k}(-1)^{k-1} = 1$

$$\therefore RHS = LHS$$

Eg: how many numbers from 1-1000 are divisible by 2,3 and 5 (note need to divide by lowest common multiple)

$$\# = \left\lfloor\frac{1000}{2}\right\rfloor + \left\lfloor\frac{1000}{3}\right\rfloor + \left\lfloor\frac{1000}{5}\right\rfloor - \left\lfloor\frac{1000}{6}\right\rfloor - \left\lfloor\frac{1000}{10}\right\rfloor - \left\lfloor\frac{1000}{15}\right\rfloor + \left\lfloor\frac{1000}{30}\right\rfloor$$

Dearangements of a set:

Eg: there area n guys, with n hats what is the number of ways no one gets the correct hat?

(answer is integer closest to $\frac{n!}{e}$)

A derangement of $\{1,2, \dots n\}$ $is\ a\ permutation\ of\ f: \{1 \dots n\} \to \{1 \dots n\}$

$$| f(x_i) \neq i \ \forall i$$

Number of derangements:

$$\#\ derangements\ of\ a\ set\ = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}$$

Proof: the number of not derangements:

$$A_i = \{permutations\ of\ f | f(i) = i\}$$
$$\therefore set\ of\ not\ derangements\ is\ (A_1 \cup A_2 \cup \dots \cup A_n)$$

$$by \frac{inclusion}{exclusion}: \ |A_{i_1} \cup \dots \cup A_{i_k}| = (n-k)!$$

$$\therefore |A_1 \cup \dots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (n-k)!$$

$$\therefore number\ of\ derangements:$$

$$\sum_{k=0}^{n} (-1)^k \frac{n!}{k!}$$

(eg: how many ways are there to send 4 notes to 4 people, so that no one gets the letter intended for them?)

$$= \sum_{k=0}^{4} \frac{(-1)^k 4!}{k!} = \frac{4!}{0!} - \frac{4!}{1!} + \frac{4!}{2!} - \frac{4!}{3!} + \frac{4!}{4!}$$

Number of surjective functions:

$$number\ of\ surjective\ functions\ from\ |X| = m, |Y| = k$$
$$\#\ surjective\ f: X \to Y$$
$$= \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^m$$

Proof: $A_i = \{f: X \to Y | f(x_i) = i\ \forall x \in X\}$

$$x = \{1..m\}; Y = \{1 \dots k\}$$
$$\therefore number\ of\ non\ surjective\ functions = |A_1 \cup A_2 \dots A_k|$$
$$= \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} (k-j)^m$$
$$\therefore number\ of\ surjecve\ functions = k^m - \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} (k-j)^m$$
$$= \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^m$$

Eg: how many ways is there to arrange 5 students to 3 teachers, so that each teacher has at least 1 student:

$$= \sum_{i=0}^{3} (-1)^i \binom{3}{i} (3-i)^5 = (3^5) - 3(2)^5 + 3(1)^5 - 0$$

Stirling numbers

For $n, k \in \mathbb{N}$,

$$S(n, k)$$

Is the number of ways to write $X = \{1 \dots n\}$ as a disjoint union of k non-empty subsets. The number of partitions of $\{1..n\}$ into k blocks.

Note:

$$S(n.n) = 1$$
$$S(n, 0) = 0\ \ (n \geq 1)$$
$$S(n, 2) = 2^{n-1} - 1$$
$$S(n, n-1) = \binom{n}{2}$$

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

Proof: for $X = \{1 \dots n\}$ we can set n by itself:

$$\dots | \dots | \dots | \dots \dots \dots .. | n = S(n-1,k-1) \ ways\ to\ partition\ n-1\ into\ k-1\ blocks$$

Or: into a set in a set with other elements:

$$\dots | \dots | \dots n \dots | \dots$$
$$= kS(n-1,k) \ \ (partitioning\ n - 1\ into\ k\ blocks, and\ k\ selections\ of\ where\ n\ could\ have\ gone)$$

m \ k	1	2	3	4	5	6	7
1	1						
2	1	1					
3	1	4	1				
4	1	11	11	1			
5	1	26	66	26	1		
6	1	57	302	302	57	1	
7	1	120	1191	2416	1191	120	1

THeorum:

Number of surjective functions

$$if\ |X| = m; |Y| = k$$
$$number\ of\ surjective\ functions\ f: X \to Y$$
$$= k!\,S(m,k)$$

Proof:

$(blob\ diagrams)$: $the\ preimages\ of\ Y\ form\ a\ partition\ of\ X\ and\ are\ disjoint\ (to\ be\ a\ function)$
$by\ definition, number\ of\ ways\ for\ this\ partition\ is\ S(m,k)$
$then\ k!\ ways\ to\ arrange\ partitions$
$\therefore k!\,S(m,k)$

Eg: arrange 5 students into 3 tutors: