

# DISCRETE MATHS AND GRAPH THEORY: MATH2069 summary

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## Chapter 1: counting principles

1. Study of non-continuous maths
2. (eg, sequences, combinatorics, induction ect)

Eg: continuous function  $y = x^2$ ; has analogous sequences  $a_n = n^2$

$$\mathbb{N} = \{0, 1, 2, \dots\}$$

For finite set  $X = \{x_1, x_2, \dots, x_n\}$ ,  $|X| = n$

The size (or cardinality) of  $X$

**Theorem: bijection principle**

If 2 sets  $X$  and  $Y$  are a bijection;  $|X| = |Y|$

3. Bijection is 1-1 correspondence, and is a way of associating to each element of  $X$  a corresponding element of  $Y$

**Injective:**

1-1:  $f: X \rightarrow Y$  if  $f(x_1) \neq f(x_2)$  for  $x_1 \neq x_2$

For each  $y \in Y$ , there is at most 1  $x \in X$  |  $f(x) = y$

**Surjective:**

'onto':  $f: X \rightarrow Y$ , if each value of  $Y$  is one of the values taken by the function (eg, if the function maps to all  $\mathbb{N}$ , then  $Y = \mathbb{N}$ )

Bijjective= surjective+injective

**Sum principle**

If  $X = A \cup B$  and  $A \cap B = \emptyset$

$$|X| = |A| + |B|$$

**Difference principle**

For any  $A \subset X$

$$|X \setminus A| = |X| - |A|$$

Preimage:  $f^{-1}(y) := \{x \in X | f(x) = y\}$

$$|X| = \sum_{y \in Y} |f^{-1}(y)|$$

Rounding notation:

$\lceil x \rceil := x$  rounded up to nearest integer

$\lfloor x \rfloor := x$  rounded down to nearest integer

$\{x\} = x$  rounded (up or down) to nearest integer

$$x - 1 < \lceil x \rceil \leq x; \quad x \leq \lfloor x \rfloor < x + 1$$

Pigeonhole principle:

$f: X \rightarrow Y$ , there must be a  $y \in Y$

$$|f^{-1}(y)| \geq \left\lceil \frac{|X|}{|Y|} \right\rceil$$

Cartesian product:

$$X \times Y := \{(x, y) | x \in X, y \in Y\}$$

Product principle:

$$|A| \times |B| \times |C| \times \dots = |A \times B \times C \dots|$$
$$|X^n| = |X|^n$$

Overcounting principle:

If  $f: X \rightarrow Y$ , and  $|f^{-1}(y)| = m$

$$\text{then } |Y| = \frac{|X|}{m}$$

Number of subsets of a set:

If  $Y \subset X = \{1, 2, \dots, n\}$

$$\therefore \text{number of subsets } Y = 2^n$$

Number of ordered selections  $(y_1 \dots y_k)$

Injective functions

Is analogous to number of injective functions

*number of injective function's from  $X \rightarrow Y$  is*

$$= n_k = \frac{n!}{(n-k)!} \quad ((-m)! = \infty)$$

Number of unordered selections:

$$= \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Binomial theorem:

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

Proof:  $(a + b)^n = (a + b)(a + b) \dots (a + b)$  ( $n$  times)

timesing everything together with the distributive law will get multiples of  $a^{n-k}b^k$  ,

Eg: aababbb ... ba ect

number of ways to choose where  $b$  sits from  $n$  places is  $\binom{n}{k}$

Properties:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\sum_{k=0}^n \binom{n}{k} = 2^n$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} = \begin{cases} 0 & n \geq 1 \\ 1 & n = 0 \end{cases}$$

$$\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$$

Multinomial coefficients:

Let  $X$  be a set with  $|X| = n$ , and  $n_1, n_2, \dots, n_m \in \mathbb{N} \mid \sum n_i = n$

let  $\binom{n}{n_1, n_2, n_3, \dots, n_m}$  = the number of ways of choosing subsets  $A_1 \dots A_m$



From  $X; |A_i| = n_i; X = A_1 \cup A_2 \dots \cup A_m$  (disjoint union)

$$\binom{n}{n_1, n_2, n_3 \dots n_m} = \frac{n!}{n_1! n_2! n_3! \dots n_m!}$$

Proof: if  $X$  lies on a line  $X = 1, 2 \dots n$

number of permutations of  $X = n!$

if we were to choose the first  $n_1$  to be in  $A_1$ , the next  $n_2$  to be in  $A_2$  etc

then the number of ways to rearrange  $X$  into subsets  $A_i$  (where the subset  $A_i$  then stays the same)

$$= \frac{n!}{n_1! n_2! \dots n_m!}$$

Eg: if there were 12 balls, and we wanted to divide them into 4 jars, of 2, 3, 5 and 2 balls in each respectively. Number of ways this can be done is  $\binom{12}{2, 3, 5, 2}$

## Multinomial theorem:

$$(a_1 + a_2 + \dots + a_m)^n = \sum_{n_1 + n_2 + \dots + n_m = n} \binom{n}{n_1, n_2, n_3 \dots n_m} a_1^{n_1} a_2^{n_2} \dots a_m^{n_m}$$

Proof analogous to binomial theorem:

Multinomial and binomial link:

$$\binom{n}{n_1, n_2, n_3 \dots n_m} = \binom{n}{n_m} \binom{n - n_m}{n_{m-1}} \dots \binom{n_1}{n_1} = \prod_{i=1}^m \binom{i}{k_i}$$

## Unordered selections with repetition

The number of unordered selections with repetition is:

$$\binom{n + k - 1}{k}$$

Proof: having chosen elements of each type in a row: eg type 1, type 2.. type n (where  $t_1 + t_2 + \dots + t_n = k$ )

We can separate these groups of each type with  $n - 1$  lines

Total objects =  $n + k - 1$

Number of ways to choose placement of each object is  $\binom{n + k - 1}{k}$

Eg: the number of ways to give 3 identical boxes to 5 people, and each person can get multiple boxes:

$$= \binom{3 + 5 - 1}{3} = \binom{7}{3}$$

## Inclusion/ exclusion principle:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Ect:

$$\begin{aligned} |A_1 \cup A_2 \cup \dots \cup A_n| &= \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| - \dots \\ &+ (-1)^{n-1} \sum |A_1 \cap A_2 \cap \dots \cap A_n| \end{aligned}$$

$$\therefore |A_1 \cup A_2 \cup \dots \cup A_n| = \sum_{k=1}^n \left[ (-1)^{k-1} \sum_{i_1, i_2, i_3, \dots, i_n=1}^n |A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_n}| \right]$$

Proof: suppose  $I$  is the subset  $I = \{1, 2, 3 \dots n\}$ , of each  $x \in A_i$

Then  $A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}$  contains  $x$  iff  $(i_1, i_2, i_3 \dots i_k \in I)$ . So, for each  $k$ ,  $x$  is counted  $\binom{|I|}{k}$

So the total number on the RHS =  $\sum_{k=1}^{|I|} \binom{|I|}{k} (-1)^{k-1} = 1$   
 $\therefore RHS = LHS$

Eg: how many numbers from 1-1000 are divisible by 2, 3 and 5 (note need to divide by lowest common multiple)

$$\# = \left\lfloor \frac{1000}{2} \right\rfloor + \left\lfloor \frac{1000}{3} \right\rfloor + \left\lfloor \frac{1000}{5} \right\rfloor - \left\lfloor \frac{1000}{6} \right\rfloor - \left\lfloor \frac{1000}{10} \right\rfloor - \left\lfloor \frac{1000}{15} \right\rfloor + \left\lfloor \frac{1000}{30} \right\rfloor$$

## Dearrangements of a set:

Eg: there are  $n$  guys, with  $n$  hats what is the number of ways no one gets the correct hat?

(answer is integer closest to  $\frac{n!}{e}$ )

A derangement of  $\{1, 2, \dots, n\}$  is a permutation of  $f: \{1 \dots n\} \rightarrow \{1 \dots n\}$   
 $| f(x_i) \neq i \forall i$

## Number of derangements:

$$\# \text{ derangements of a set} = \sum_{k=0}^n (-1)^k \frac{n!}{k!}$$

Proof: the number of not derangements:

$A_i = \{\text{permutations of } f \mid f(i) = i\}$   
 $\therefore \text{set of not derangements is } (A_1 \cup A_2 \cup \dots \cup A_n)$

by  $\frac{\text{inclusion}}{\text{exclusion}}$ :  $|A_{i_1} \cup \dots \cup A_{i_k}| = (n - k)!$

$$\therefore |A_1 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} (n - k)!$$

$\therefore \text{number of derangements:}$

$$\sum_{k=0}^n (-1)^k \frac{n!}{k!}$$

(eg: how many ways are there to send 4 notes to 4 people, so that no one gets the letter intended for them?)

$$= \sum_{k=0}^4 \frac{(-1)^k 4!}{k!} = \frac{4!}{0!} - \frac{4!}{1!} + \frac{4!}{2!} - \frac{4!}{3!} + \frac{4!}{4!}$$

## Number of surjective functions:

number of surjective functions from  $|X| = m, |Y| = k$

# surjective  $f: X \rightarrow Y$

$$= \sum_{i=0}^k (-1)^i \binom{k}{i} (k-i)^m$$

Proof:  $A_i = \{f: X \rightarrow Y \mid f(x_i) = i \forall x \in X\}$

$x = \{1..m\}; Y = \{1..k\}$

$\therefore$  number of non surjective functions =  $|A_1 \cup A_2 \dots A_k|$

$$= \sum_{j=1}^k (-1)^{j-1} \binom{k}{j} (k-j)^m$$

$\therefore$  number of surjective functions =  $k^m - \sum_{j=1}^k (-1)^{j-1} \binom{k}{j} (k-j)^m$

$$= \sum_{j=0}^k (-1)^j \binom{k}{j} (k-j)^m$$

Eg: how many ways is there to arrange 5 students to 3 teachers, so that each teacher has at least 1 student:

$$= \sum_{i=0}^3 (-1)^i \binom{3}{i} (3-i)^5 = (3^5) - 3(2^5) + 3(1)^5 - 0$$

## Stirling numbers

For  $n, k \in \mathbb{N}$ ,

$$S(n, k)$$

Is the number of ways to write  $X = \{1..n\}$  as a disjoint union of  $k$  non-empty subsets. The number of partitions of  $\{1..n\}$  into  $k$  blocks.

Note:

$$S(n, n) = 1$$

$$S(n, 0) = 0 \quad (n \geq 1)$$

$$S(n, 2) = 2^{n-1} - 1$$

$$S(n, n-1) = \binom{n}{2}$$

$$S(n, k) = S(n - 1, k - 1) + kS(n - 1, k)$$

Proof: for  $X = \{1 \dots n\}$  we can set  $n$  by itself:

$\dots | \dots | \dots | \dots \dots \dots | n = S(n - 1, k - 1)$  ways to partition  $n - 1$  into  $k - 1$  blocks

Or: into a set in a set with other elements:

$\dots | \dots | \dots n \dots | \dots$   
 $= kS(n - 1, k)$  (partitioning  $n - 1$  into  $k$  blocks, and  $k$  selections of where  $n$  could have gone)

		k						
	1	2	3	4	5	6	7	
1	1							
2	1	1						
3	1	4	1					
m	4	1	11	11	1			
5	1	26	66	26	1			
6	1	57	302	302	57	1		
7	1	120	1191	2416	1191	120	1	

Theorem:

## Number of surjective functions

if  $|X| = m; |Y| = k$

number of surjective functions  $f: X \rightarrow Y$

$$= k! S(m, k)$$

Proof:

(blob diagrams): the preimages of  $Y$  form a partition of  $X$  and are disjoint (to be a function) by definition, number of ways for this partition is  $S(m, k)$

then  $k!$  ways to arrange partitions

$$\therefore k! S(m, k)$$

Eg: arrange 5 students into 3 tutors: