# MATH2970 optimisation and financial maths lecture notes

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## Contents

# OPTIMISING DIFFERENTIALE FUNCTIONS

## CHAPTER 1: OPTIMISING DIFFERENTIABLE FUNCTIONS

Examples: physics, chemical reactions, scheduling, manufacturing.

### *Eg: Minimising Surface area of can*

Start by modelling the can as a cylinder:

$$V = \pi r^2 h = 375mL$$
$$S = 2\pi r + 2\pi r^2$$
$$h = \frac{V}{\pi r^2}$$
$$\Longrightarrow S(\mathrm{r}) = \frac{2\pi}{r} + 2\pi r^2$$
$$\therefore differentiate\ ect$$

$$\therefore r = \left(\frac{V_0}{r\pi}\right)^{\frac{1}{3}}$$

But: this gives $r \approx 3.91; h = 7.82$. So why is this different to the ACTUAL size of a can?

Modelling in incorrect (eg- S has no width, indentation at bottom ect)

$$industrial\ paramenters\colon\ d_{side} = 0.0104cm$$
$$d_{top} = 0.0236cm$$
$$d_{bottom} = 0.0203cm$$

$$\therefore \bar{S}(h,r)\ (not\ an\ area) = 2\pi rhd_{side} + \pi r^2 d_{bottom} + \pi r^2 d_{top}$$
$$\therefore \bar{S}(r) = 2\pi r\left(\frac{V}{\pi r^2}\right)d_{side} + \pi r^2\left(d_{bottom} + d_{top}\right) = \frac{2V}{r}d_{side} + \pi r^2(d_{bottom} + d_{side})$$
$$\frac{d\bar{S}}{dr} = -\frac{2V}{r^2}d_{side} + 2\pi r(d_{bottom} + d_{side}) = 0$$

### Mathematical optimisation:

Given an **objective function**, $f\colon \mathbb{R}^n \to \mathbb{R}\quad (\text{scalar function})$

And a **feasible region**: $\Psi$

And **optimisation problem** is the problem of finding an $x^* \in \mathbb{R}^n$ that solves:

$$\min_{x\in\mathbb{R}^n} f(x)\ \ |x \in \Psi\ \ or\ \ \max_{x\in\mathbb{R}^n} f(x)\ \ |x \in \Psi$$

### Optimisation of differentiable functions of one variable

Some scenarios:

$$f(x)\ is\ constant\ for\ x \in [a,b]$$
$$\Longrightarrow all\ x \in [a,b]\ optimised$$

$$f(x)\ is\ linear\ for\ x \in [a,b]$$
$$\therefore optimised\ points\ on\ boundary$$

$$f(x)\ has\ unique\ global\ extremity\ in\ interior:$$

$$f(x)\ has\ multiple\ local\ maximum\ or\ minimum$$
$$must\ use\ computers\ and\ find\ an\ algorithm\ to\ solve\ it$$

## Global minimum and maximum

Definition: a point $x^*$ is a **global minimum** if $f(x^*) \leq f(x)\ \forall x \in \Psi$

Definition: a point $x^*$ is a **local minimum** if there is a neighbourhood $N$ of $x^* | f(x^*) \leq f(x) \forall x \in N$

### *Identifying local extremities of $f(x)$*

1. First derivative test $f'(x^*) = 0$ (could be min, max or inflexion), only necessary condition for the existence of optimal
2. Sufficient condition can be established using higher order derivatives:
   - $f'(x^*) = 0;\ f''(x^*) < 0$: local maximum
     But: eg this would not mind max of $-x^4$
   - If $f'(x^*) = f''(x^*) = \cdots = f^{2m-1}(x^*) = 0$ and
     $$f^{2m}(x^*) < (>) 0, x^*\ is\ maximum (minimum)$$
   - If $f^1(x^*) = \cdots = f^{2m}(x^*) = 0$, and $f^{2m-1}(x^*) \neq 0$, then $x^*$ is a point of inflection

### *Finding global extremity:*

Now we can test for global extremity:

$$\min\{f(a), f(b), f(x_1^*), f(x_2^*) \ldots, f(x_k^*)\}$$

(or max)

## Revision of solving linear equations:

Eg:

$$Ax = b$$
$$x = A^{-1}b$$

## Pivot operation algorithm:

Eg solve:

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix}$$

1. Decide on a **pivot element**, $a_{ij} \neq 0$
2. Divide row $i$ by $a_{ij} \neq 0$ (in our lecture, $a_{ij}$ are large enough to not amplify errors (that it may in a computer))
3. Transform all other rows of $a_{kj}$ $(k \neq i)$ by adding suitable multiples of row $i$

Eg: in tableaux form

| $x_1$ | $x_2$ | $x_3$ | $b$ |
|---|---|---|---|
| 1 | -1 | 1 | -2 |
| 2 | 1 | -1 | 5 |
| -1 | 2 | 3 | 0 |

| $x_1$ | $x_2$ | $x_3$ | $b$ |
|---|---|---|---|
| 1 | -1 | 1 | -2 |
| 0 | 3 | -3 | 9 |
| 0 | 1 | 4 | -2 |

| $x_1$ | $x_2$ | $x_3$ | $b$ |
|---|---|---|---|
| 1 | 0 | 5 | -4 |
| 0 | 0 | -15 | 15 |
| 0 | 1 | 4 | -2 |

$$x_3 = -1; x_2 = -2 - 4(-1) = 2; x_1 = -4 - 5(-1) = 1$$

$$\therefore \boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

## Transformation of linear functions with Gaussian Jordan elimination

### *Basic/nonbasic variables*

If we were given:

$$\begin{pmatrix} 1 & -1 & 1 & -1 & 0 \\ 2 & 1 & -1 & 0 & 1 \\ -1 & 2 & 3 & 1 & 2 \end{pmatrix}\boldsymbol{x} = \begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix}$$

As this system has infinite solutions, we can simplify a solution $Z = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5x_5 + c_0$ into the form:

$$Z = Ax_4 + Bx_5 + C$$

(as $x_1, x_2$ $and$ $x_3$ can be expressed in terms of $x_4$ and $x_4$

In this case: the variables which have a unique solution are known as **non-basic**, whereas the one's which do not $(x_4, x_5)$ are called **basic**

Eg: the system above simplifies to:

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{8}{15} & \frac{2}{3} \\ 0 & 0 & 1 & -\frac{2}{15} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

So: $x_1 = 1 + \frac{1}{3}x_4 - \frac{1}{3}x_5;\quad x_2 = 2 - \frac{8}{15}x_4 - \frac{2}{3}x_5;\ x_3 = -1 + \frac{2}{5}x_4 - \frac{1}{3}x_5$

# LINEAR PROGRAMMING

## CHAPTER 2: LINEAR PROGRAMMING

- The term **programming** means planning/logistics (not computing)
    - Used for : allocating limited resources among competing activities in optimal way
    - Selecting the level of certain activities that compete for limited resources to optimise some objective function
        - Eg:
            - Resource allocation
            - Portfolio selection
            - Transportation
            - Agriculture
            - Manufacturing

### Standard LP Problem:

I. Maximise $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

II. Subject to:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & \dots & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} \leq \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix}$$

III. With: $x_1, x_2, \dots x_n \geq 0$

OR:

Maximise $Z = \boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x}$

Subject to $A\boldsymbol{x} \leq \boldsymbol{b}$

And $\boldsymbol{x} \geq \boldsymbol{0}$

I. $Z$ is the **objective function.** It is a linear function of the **decision variables** $(x_1, x_2, \dots, x_n)$. The constants $(c_1, c_2, \dots, c_n)$ are the **cost coefficients**. The increase in $Z$ for unit increase in $x_k$ is $c_k$.

II. This part states the **linear constraints** of the problem. The coefficient matrix is the **constraint matrix.** In standard LP problems, all elements of the **resource vector** $(b_1, b_2, \dots, b_n)$ are assumed to be non-negative.

III. The final part of the LP problem is the **positivity condition**: of the decision variables $(x_1, x_2, \dots, x_n)$

Any $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ that satisfy II and III are **feasible solutions**, and lie in a closed region in the decision space, called the **feasible region:** $\Psi$**.** The decision space is always non-empty as $(0,0, \dots 0)$ is always feasible.

Any $\boldsymbol{x}$ not in the feasible region is **infeasible**.

A feasible solution of $\boldsymbol{x}$ which maximimes the objective function $Z$ is the **optimal solution.** Denoted $\boldsymbol{x}^*$.

As the objective function is Linear (in standard LP problems), the maximum and minimum of $Z$ must lie on the boundary of the feasible region.

*Example of LP problem:*

| | Resources $(P_4)$ needed percent of product | | |
|---|---|---|---|
| | Product | | |
| "competing" sites | white | blue | Amount of resources available |
| $RV_1$ | 1 | 0 | 4 |
| $RV_2$ | 3 | 2 | 18 |
| $RV_3$ | 0 | 2 | 12 |
| Objective function $Z$ | 3 | 5 | |

$$\therefore LP\ problem\ is:$$

If $x_1$ is the number of white units

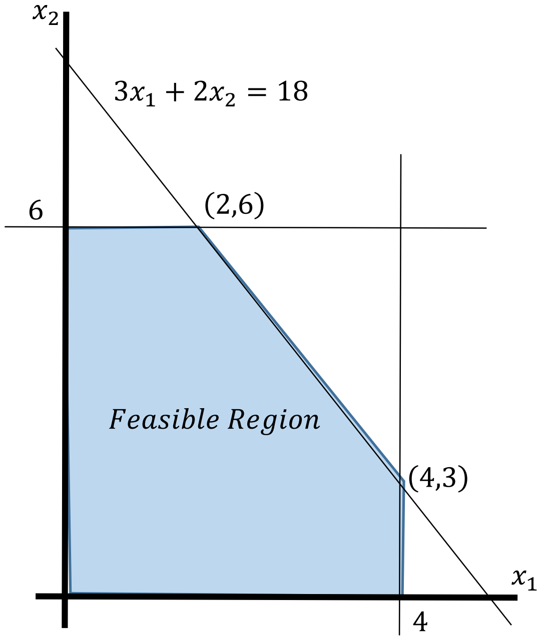
$x_2$ is the number of blue units:

$$\therefore Maximise: Z = 3x_2 + 5x_2$$

Subject to:

$$x_1 \leq 4$$
$$3x_1 + 2x_2 \leq 18$$
$$2x_2 \leq 12$$

And $x_1, x_2 \geq 0$




## To maximise:

Either:

1. Look at slope of contour lines of constant $z$ → typically meets at corner points

   Make equation $x_2 = -\frac{3}{5}x_1 + \frac{1}{5}z$: keeping $z$ constant; then shift line up until you reach the end:

   - Will most likely be a point, but could meet a boundary if line is parallel to boundary (in which case they are all the most optimal)

2. Compute value of $Z$ at corner points

## Notes on the feasible region:

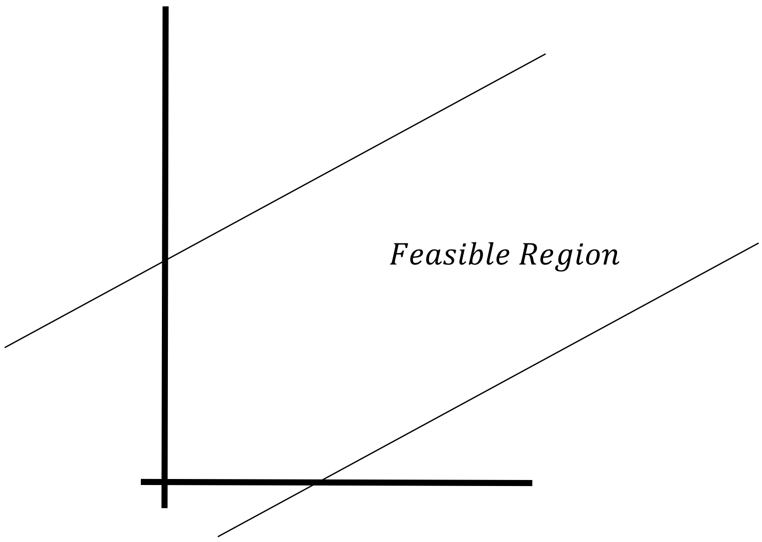
1. Feasible region may not exist (inconsistent constraints, eg $x_2 < 0$)
2. Feasible region may be unbounded

   Eg.

May be more than one optimal solution (eg optimal lies on a boundary):

Eg: $Z = 6x_1 + 4x_2$:, $with\ 3x_1 + 2x_2 \leq 18$

$$Z^* = \left(t, 9 - \frac{3}{2}t\right)$$




However- the MINIMUM will still exist

3. Feasible region is **convex**
    - A set $R$ is convex if, $\forall x \in R$, and scalars $\lambda \in [0,1]$, $z = \lambda x + (1-\lambda)y$ satisfies $z \in R$. (if I take any 2 lines on the boundary, and draw a line between, then all points on the line lie in the feasible region)

## The Simplex Algorithm: (graphically)

1. Initialisation: start at a **FCP (feasible corner point)**, with objective function value $Z$.
2. Iteration Step: Move to an adjacent FCP with the best potential of $Z$ increase
3. Stopping rule: Stop at $FCP^\star$ if its $Z^\star$ is $\geq$ the $Z$ valuesof all its adjacent FCP's.

*Eg: Drug problem*

in the drug problem above:

1. Start at $F_{1|_{x_2=0}^{x_1=0}}$ with $Z = 0$
2. 
    - Move to $F_5|_{x_2=6}^{x_1=0}$, as $Z = 3x_1 + 5x_2$ increases sharpest in $x_2$ direction (as $5 > 3$) with $Z = 30$
    - Move to $F_4$ $((x_1, x_2) = (2,6)$; with $Z = 36$
    - $F_5$, $(4,3)$, $Z = 27$
3. Stop at $F_4^\star = (2,6)$: $Z^\star = 36$

Corner points are intersections of constraints:

Constraints are hyperplanes in $\mathbb{R}^n$, solutions to $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$

$\therefore$ in LP, a FCP needs $n$ of $(n+m)$ constaints in $\mathbb{R}^n$

## Total number of corner points

Therefore, the total number of corner points is $\binom{m+n}{n}$ corner points (but some are infeasible)

## Algebraic Representations of Corner Points

For each constraint in the subject to; introduce a "**slack variable**". Equal to the difference between the LHS and RHS of the constraint:

At a corner point:

- $n$ variables are 0 (non basic variable)
- $m$ variables are non zero (basic variables)

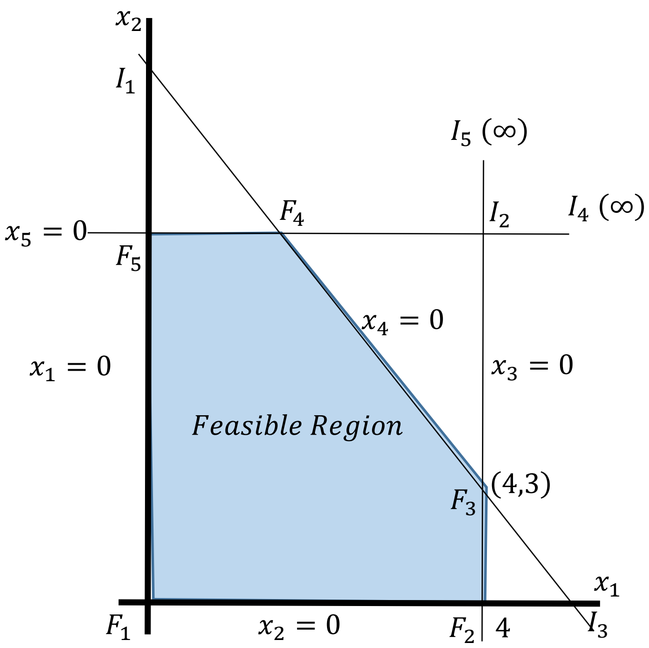
### Eg:

If:

$$x_1 \leq 4; \quad x_3 = 4 - x_1 \geq 0$$
$$3x_1 + 2x_2 \leq 18; \quad x_4 = 18 - 3x_2 - 2x_2 \geq 0$$
$$2x_2 \leq 12; \quad x_5 = 12 - 2x_2 \geq 0$$

Now: the boundary of the feasible regions are:

$$x_1 = x_2 = \cdots = x_5 = 0$$




$$FCP = F_1 \rightarrow F_5 \text{ are } \textbf{feasible}$$
$$I_1 \rightarrow I_5 \text{ are } \textbf{infeasible}$$

## Adjacent corner points

Two corner points are **adjacent** if they differ in exactly 1 non-basic variable (or, equivalenty, one basic variable)

(so- in the example beflow: $F_1(0,0)$ is adjacent to $F_2(4,0)$ and also $I_3(6,0)$ and $F_5(0,6)$ and $I_3(0,9)$ using this definition. BUT: to get to infeasible corner points, we must cross a feasible boundary)

- A corner point $C$ is adjacent to ALL cornerpoints on the boundaries passing through $C$ (not just FCP)

## Moving between CP's

To move from one CP to an adjacent CP, **ONE** basic variable is replaced by a non-basic variable

We say a variable "**enters**" or **"leaves"** the basis

So in the Drug problem: $F_1 \to F_5 \to F_4 \to F_3$

$$F_1|(0,0,4,18,12) \to F_5\ (0,6,4,6,0)$$

so $x_2$ enters the basis, and $x_5$ left the basis

## The simplex algorithm (algebraically)

Aim: Move from an FCP to an adjacent FCP with largest potential increase in the objective function $Z$. Find theoretically the FCP which maximises $Z$ $(finds\ Z^*)$

## Standard LP problem:

Maximise $Z = \boldsymbol{c}^T\boldsymbol{x}$

Subject to: $A\boldsymbol{x} \leq \boldsymbol{b}$ $(\boldsymbol{x} \in \mathbb{R}^n; \boldsymbol{b} > \boldsymbol{0})$

With $\boldsymbol{x} \geq \boldsymbol{0}$

*1. Initialisation:*

Choose a feasible solution

- Write the LP problem in tableu form:

| $Z$ | $x_1$ | $x_2$ | … | $x_n$ | $x_{n+1}$ | … | $x_{n+m}$ | $b$ |
|---|---|---|---|---|---|---|---|---|
| 1 | $-c_1$ | $-c_2$ | … | $-c_n$ | 0 | | 0 | 0 |
| 0 | $a_{11}$ | $a_{12}$ | … | 1 | 0 | … | 0 | $b_1$ |
| 0 | $a_{21}$ | $a_{22}$ | … | 0 | 1 | … | 0 | $b_2$ |
| … | … | … | … | … | … | … | … | … |
| 0 | $a_{m1}$ | $a_{m2}$ | … | 0 | 0 | … | 1 | $b_m$ |

Set decision variables $(x_1, x_2, \dots x_n) = \boldsymbol{0}$ and slack variables $(x_{n+1}, x_{n+2}, \dots, x_{n+m}) = (b_1, b_2, \dots b_m)$

(Is feasible solution as $b_i > 0$ so $x_{n+i} > 0$)

In Matrix form, this is represented at:

Maximise $Z$:

### Matrix form of simplex:

$$\begin{bmatrix} 1 & -\boldsymbol{c}^T & \boldsymbol{0}_{row} \\ \boldsymbol{0}_{column} & \boldsymbol{A} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} Z \\ \boldsymbol{x} \\ \boldsymbol{x_S} \end{bmatrix} = \begin{bmatrix} 0 \\ \boldsymbol{b} \end{bmatrix}$$

$$where: \boldsymbol{c} = \begin{pmatrix} c_1 \\ c_2 \\ \dots \\ c_n \end{pmatrix}; \boldsymbol{x} = (x_1, x_2, \dots, x_n); \boldsymbol{x_s} = slack\ variables$$

For Drug Problem

| $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $b$ |
|---|---|---|---|---|---|---|
| 1 | $-3$ | $-5$ | 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 1 | 0 | 0 | 4 |
| 0 | 3 | 2 | 0 | 1 | 0 | 18 |
| 0 | 0 | 2 | 0 | 0 | 1 | 12 |

$$\therefore initial\ values: (x_1, x_2, \dots x_5) = (0,0,4,18,12)\ \big(2\ non\ basic\ (x_! \ x_2), 3\ basic\ \ (x_3\ x_4\ x_5)\big)$$

### 2. *Iteration step*

We need a criteria for:

a) Which of the non-basic variables will enter the basis

b) Which of the basic variables will leave the basis.

(i.e.- which adjacent FCP we should move to)

### Largest coefficient rule (entering basis):

a) Which of the non-basic variables should enter?

For $Z = \sum_{i=1}^{n} c_i x_i$; a good choice for which adjacent variable we should use is in the direction with the greatest cost coefficient ($c_i$) – this MIGHT yield the largest increase in $Z$ (and so the least number of steps); but it also could not (the only way we can tell is by calculating it)

So: increase the non basic variable $x_e = 0$ to $x_e > 0$; where $c_e = \max\{c_i's\}$

Eg: for the drug problem:

$Z = 3x_1 + 5x_2$; move in the direction of $x_2$ ($x_2$ should enter the basis)

### Rule for exiting basis:

b) Which of the basic variables should leave?

Take the variable $x_\ell$ which will become zero first upon increasing $x_e$. (this finds the 'immediate neighbours')

Eg:

- Graphically:
  - $x_5 = 0$ is reached before $x_4 = 0$ on the boundary $x_1 = 0$ when varing $x_2$
- $Algebraically$:
  - We have $x_1 = 0$; and we're varying $x_2$
    Constaints are:
    $$x_3 = 4:$$
    $$2x_2 + x_4 = 18 \rightarrow x_4 = 18 - 2x_2$$
    $$2x_2 + x_5 = 12 \rightarrow x_5 = 12 - 2x_2$$
    So: if we vary $x_2 > 0$: $x_5$ will become 0 before $x_4$ will (and $x_3$ is unaffected by $x_2$) :
    - SO $x_5$ SHOULD LEAVE THE BASIS‼

RULE:

Choose the $x_\ell$ such that $\frac{b_i}{a_{ie}}$ is minimised for $i = \ell$

## Example of iteration: Drug problem

| Basis | $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $b$ | $\frac{b_i}{a_{i2}}$ |
|---|---|---|---|---|---|---|---|---|
| $Z$ | 1 | $-3$ | $-5$ | 0 | 0 | 0 | 0 | $-$ |
| $x_3$ | 0 | 1 | 0 | 1 | 0 | 0 | 4 | $-$ |
| $x_4$ | 0 | 3 | 2 | 0 | 1 | 0 | 18 | $\frac{18}{2} = 9$ |
| $x_5$ | 0 | 0 | 2 | 0 | 0 | 1 | 12 | $\frac{12}{2} = 6$ |

$$\therefore as\ 6\ is\ min: choose\ x_5\ to\ leave\ rather\ than\ x_4$$

Use Gaussian elimination to eliminate the $x_2$ column:

| Basis | $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $b$ | $\frac{b_i}{a_{i2}}$ |
|---|---|---|---|---|---|---|---|---|
| $Z$ | 1 | $-3$ | $-5$ | 0 | 0 | 0 | 0 | $-$ |
| $x_3$ | 0 | 1 | 0 | 1 | 0 | 0 | 4 | 4 |
| $x_4$ | 0 | 3 | 0 | 0 | 1 | $-1$ | 6 | 2 |
| $x_5$ | 0 | 0 | 1 | 0 | 0 | $\frac{1}{2}$ | 6 | $-$ |

Min =2: so choose $F_4$ rather than $I_2$ (now $Z$ is ixpressed in terms of $x_1$ and $x_5$)

- $x_1$ should enter the basis as it has the largest MODIFIED cost coefficient ($\bar{c_1} = -(-3\,) = 3$)

So: GJ elimination on $x_1$ column and $x_4$ row:

| Basis | $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $b$ |
|---|---|---|---|---|---|---|---|
| $Z$ | 1 | 0 | 0 | 0 | 1 | $\frac{3}{2}$ | 36 |
| $x_3$ | 0 | 0 | 0 | 1 | $-\frac{1}{3}$ | $\frac{1}{3}$ | 2 |
| $x_4$ | 0 | 1 | 0 | 0 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 2 |
| $x_5$ | 0 | 0 | 1 | 0 | 0 | $\frac{1}{2}$ | 6 |

SO:

$$Z = -x_4 - \frac{3}{2}x_5 + 36$$

3. *Stopping rule:*

When all modified cost coefficients $\overline{c_i} \leq 0$ (when all $x$ entroes in the $Z$ row are positive_; one cannot move to an adjacent FCP without decreasing $Z$, and the tableu is optimal

## Summary of standard LP form:

Maximise $Z$

Each $b_i \geq 0$

Constraints $\leq 0$

Variables $\geq$

## Possible problems

### Tie breaking rule for cost coefficients:

Eg:

| $Basis$ | $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $RHS$ |
|---|---|---|---|---|---|---|---|
| $Z$ | 1 | 0 | $-3$ | $-3$ | 0 | 0 | 20 |

Largest coefficient rule leads to an ambiguity: either choice will work (cannot really predict which is better). It is not predictable which choice will give the potentially quickest solution.

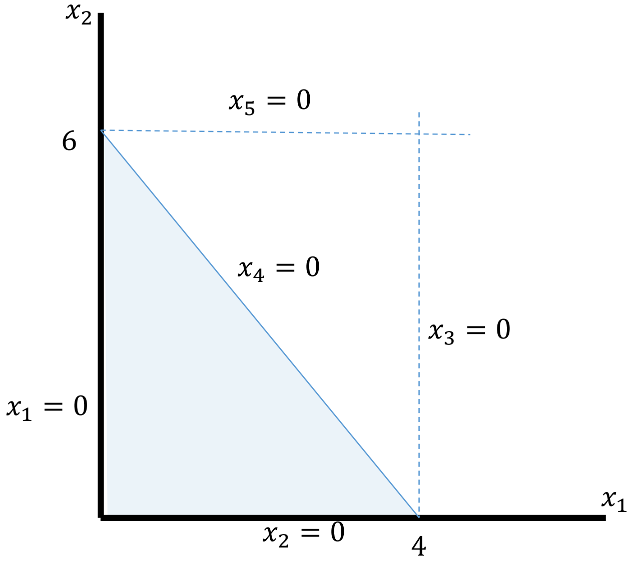
### Tied ratios:

Eg.

Consider the feasible problem: $Z = 3x_1 + 5x_2$

$$with\ x_1 \leq 4$$
$$3x_1 + 2x_2 \leq 12$$

$$2x_2 \leq 12$$




There are 3 $x_i' s$ intersecting at each FCP

Algebraically:

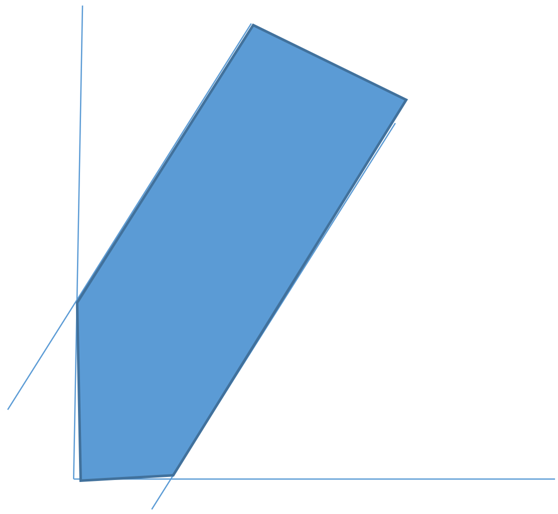
| $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $b$ | Ratio: |
|---|---|---|---|---|---|---|---|
| 1 | −3 | −5 | 0 | 0 | 0 | − | − |
| 0 | 1 | 0 | 1 | 0 | 0 | 4 | − |
| 0 | 3 | 2 | 0 | 1 | 0 | 12 | 6 |
| 0 | 0 | 2 | 0 | 0 | 1 | 12 | 6 |

NOTICE: Equal ratio in Ratio column, so take either, $x_1$ or $x_2$, no way to tell which is better.

## No leaving variable:

(i.e- unbounded solution)

Graphically:



Eg:

Max $Z = 3x_1 + 5x_2$

Such that:

$$-x_1 + x_2 \leq 4$$

$$x_1 - x_2 \leq 2$$

With $x_i \geq 0$

| $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $b$ | Ratio: |
|---|---|---|---|---|---|---|
| 1 | $-3$ | $-5$ | 0 | 0 | $-$ | $-$ |
| 0 | 1 | 1 | 1 | 0 | 4 | 4 |
| 0 | 1 | $-1$ | 0 | 1 | 2 | $-$ |

$x_3$ *leaves*, $x_2$ *enters*

| $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $b$ | Ratio: |
|---|---|---|---|---|---|---|
| 1 | $-8$ | 0 | 5 | 0 | 20 | $-$ |
| 0 | $-1$ | 1 | 1 | 0 | 4 | $-$ |
| 0 | 0 | 0 | 1 | 1 | 6 | $-$ |

No variable to leave basis!

$$\therefore \text{solution is unbounded}$$

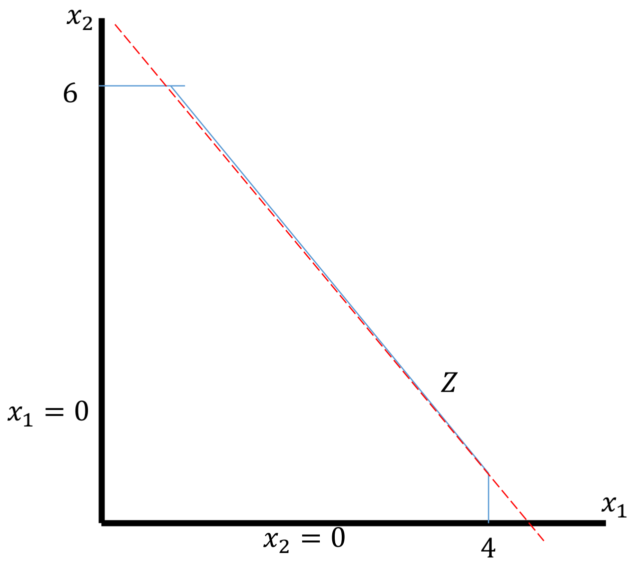
## Multiple optimal solutions:

$Z$ is parallel to an $x$

Eg:

$$Maximise: \ z = 6x_1 + 4x_2$$

$$x_1 \leq 4;$$
$$3x_1 + 2x_2 \leq 18$$
$$2x_2 \leq 12$$




$Z = 36$ is optimal for any $x_4 = 0$

| $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $b$ | Ratio: |
|---|---|---|---|---|---|---|---|
| 1 | $-6$ | $-4$ | 0 | 0 | 0 | 0 | – |
| 0 | 1 | 0 | 1 | 0 | 0 | 4 | 4 |
| 0 | 3 | 2 | 0 | 1 | 0 | 18 | 6 |
| 0 | 0 | 2 | 0 | 0 | 1 | 12 | 6 |

| $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $b$ | Ratio: |
|---|---|---|---|---|---|---|---|
| 1 | 0 | $-4$ | 6 | 0 | 0 | 24 | – |
| 0 | 1 | 0 | 1 | 0 | 0 | 4 | – |
| 0 | 0 | 2 | $-3$ | 1 | 0 | 6 | 3 |
| 0 | 0 | 2 | 0 | 0 | 1 | 12 | 6 |

| $Z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $b$ | Ratio: |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 2 | 0 | 36 | |
| 0 | 1 | 0 | 1 | 0 | 0 | 4 | |
| 0 | 0 | 1 | $-\frac{3}{2}$ | $\frac{1}{2}$ | 0 | 3 | |
| 0 | 0 | 0 | 3 | -1 | 1 | 6 | |

$$\therefore \, Optimal \; as \; all \; Z \; is \; positive:$$

$$Z = 36 - 2x_4$$

∴ Optimal solution for $x_4 = 0$:

Subbing in $x_3 = t$:

$$\therefore x_1 = 4 - t; x_2 = 3 + \frac{3}{2}t; x_5 = 6 - 3t$$

$$\therefore t, x_1, x_2, x_5 \geq 0$$

Solving for $t$:

$$\therefore t \in [0,2]$$

$$\therefore \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 4 - t \\ 3 + \frac{3}{2}t \\ t \\ 0 \\ 6 - 3t \end{pmatrix} \; with \; t \in [0,2]$$

THEREFORE:

Assigne to each optimal non-basic variable with a zero modified cost coefficient an arbitrary parameter $t_i$

## Summary of the Simplex algorithm for STANDARD LP problems:

0. Check LP problem is in **standard form**
   - ➔ Maximisation problem
   - ➔ Check that each $b_i \geq 0$
   - ➔ Constraints are all $\leq$ (ie. $ax + bx_2 + \cdots \leq b_i$)
   - ➔ Positivity constraints $x_i \geq 0$
   - Modify each one that does not follow
1. For each $\leq$ constraint, introduce the **slack variable** $x_{n+m} \geq 0$
2. As the initial feasible corner point solution: Set all decision variables (original set of variables problem is formulated in) to 0
3. At each iteration:
   a. **The entering basic variable has the most negative cost coefficient in the $Z$ row**
   b. **The leaving basic variable $x_i$ corresponds to the row $i_0$ such that $\min_i \frac{b_i}{a_{ij}} = \frac{b_{i_0}}{a_{i_0j}}$ for $a_{ij} > 0$, where $j$ is the index corresponding to the entering basic variable.**
   c. **Use Gauss-Jordan elimination to reduce $a_{i_0j} = 1, a_{ij} = 0$, for $i \neq i_0$**
4. Repeat step 3 above until all modified cost coefficients in the $Z$ row are $\geq 0$, then stop and read off the optimal solution

## Efficiency of Simplex algorithm:

Empirical evidence indicates that for $m$ constraints, the simplex algorithm takes approximates $1.5m - 2m$ iterations to converge to optimal solution.

## Klee-Mitty problem:

Worst possible convergence of simplex algorithm: traverses all FCP to come to the answer in $2^m - 1$ iterations

## Adapting the simplex algorithm to non-standard problems:

### Minimising the objective function:

To minimise: $Z = \sum_i c_i x_i$, define a new objective function:

$$\hat{Z} = -Z$$

Then:

$$\min Z = -\max \hat{Z}$$

#### *General minimisation:*

For $f: \mathbb{R}^n \to \mathbb{R}$,

$$\min_{x_1, x_2, \dots, x_n} f(x_1, x_2, \dots, x_n) = -\max_{x_1, x_2, \dots x_n} [-f(x_1, x_2, \dots, x_n)]$$

### Negative resource elements:

In the standard LP problem, we required all resource elements $b_j$ to be non-negative.

Suppose that $b_j = -b < 0$. i.e. $a_1 x_1 + a_2 x_2 + \cdots \leq -b$

This is equivalent to:

$$-a_1 x_1 - a_2 x_2 - \cdots \geq b$$

→ So we can assume a resource element is always non negative. IF we can modify the simplex algorithm to include ≥ constaints.

### Greater than or equal to constrains.

If: $a_1 x_1 + a_2 x_2 + \cdots \geq b \geq 0$

Introduce a **surplus variable $x_{n+1} \geq 0$** such that

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n - x_{n+1} = b$$

### Negative decision variable:

If $x_k \leq 0$: introduce a new variable $\hat{x}_k = -x_k$, then $x_k \leq 0 \Leftrightarrow \hat{x}_k > 0$

### Decision variable $\geq k$:

If $x \geq k > 0$; introduce $\hat{x} = x - k \geq 0$

### Unrestricted Decision variable:

If $x_k$ is unrestricted in sign, introduce **two** new variables $\hat{x}_k \geq 0$ AND $\hat{\hat{x}}_k \geq 0$, and let $x_k = \hat{x}_k - \hat{\hat{x}}_k$

### Equality constraints:

$$\boldsymbol{a}^{\boldsymbol{T}}\boldsymbol{x} = a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \geq 0$$

Several approaches:

- Eliminate a variable: and the equality constraints will disappear

Eg: $x_n = \frac{b}{a_n} - \frac{a_1}{a_n}x_1 - \cdots - \frac{a_{n-1}}{a_n}x_n$

- Use the fact that $\boldsymbol{a}^{\boldsymbol{T}}\boldsymbol{b} = \boldsymbol{0} \Leftrightarrow \boldsymbol{a}^{\boldsymbol{T}}\boldsymbol{b} \geq \boldsymbol{0}\ and\ \boldsymbol{a}^{\boldsymbol{T}}\boldsymbol{b} \leq \boldsymbol{0}$
- Use an artificial variable

## Finding an initial FCP solution

Finding and initial FCP solution can be as hard as finding the optimal solution itself.

- Recall for standard LP problem: $\max Z = \boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x} | \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, with\ \boldsymbol{x} \geq \boldsymbol{0}$
  - To get an initial FCP solution,we introduce slack variables $\boldsymbol{x}_{\boldsymbol{s}} \geq \boldsymbol{0}$. We set $\boldsymbol{x} = \boldsymbol{0}$ and $\boldsymbol{x}_{\boldsymbol{s}} \geq \boldsymbol{0}$

This will fail if there is an equality of $\geq$ constraint in a non standard LP problem

1) $\boldsymbol{a}^{\boldsymbol{T}}\boldsymbol{x} = \boldsymbol{b}: \boldsymbol{x} = \boldsymbol{0} \rightarrow \boldsymbol{a}^{\boldsymbol{T}}\boldsymbol{x} = \boldsymbol{0} \neq \boldsymbol{b}.$ So doesn't work
2) $\boldsymbol{a}^{\boldsymbol{T}}\boldsymbol{x} \geq \boldsymbol{b} > \boldsymbol{0}: x = 0 \rightarrow 0 \geq b > 0$. Doesn't work

**Eg:**

$$\begin{aligned}
&\max Z = 3x_1 + 5x_2 \\
&with\ x_1 \leq 4 \\
&3x_1 + 2x_2 \geq 18 \\
&2x_2 \leq 12 \\
&with\ x_1, x_2, x_3 \geq 0
\end{aligned}$$

Introduce slack variables: $x_3, and\ x_5\ \ and\ a\ surplus\ variable\ x_4.$: so the constraints become

$$\begin{aligned}
&x_1 + x_3 = 4 \\
&2x_1 + 2x_2 - x_4 = 18 \\
&2x_2 + x_5 = 12
\end{aligned}$$

Stragetry is to introduce an **"artificial variable"** $\bar{x}_6$ (bar indicates artificial),

*Note on $=$ or $\geq$ constraints for artificial variables $\bar{x}_k$*

- We introduce an artificial variable for each $\geq or =$ constraint, on top of the surplus variable (which is has a negative coefficient)

The constraints become:

$$x_1 + x_3 = 4$$
$$2x_1 + 2x_2 - x_4 + \bar{x}_6 = 18$$
$$2x_2 + x_5 = 12$$
$$with\ x_{1\to 6} \geq 0$$

Our initial FCP is given by setting:

$$x_{decision} = 0 \to x_1, x_2 = 0$$
$$x_{surplus} = 0 \to x_4 = 0$$
$$x_{slack} = RHS$$
$$x_{artificial} = RHS$$

(note: so the $-x_4 + \bar{x}_6 = (0) + 18 = 18$) so it holds

Note: we do not recover the initial $\geq$ constraint unless $\bar{x}_6 \to 0$

## Problem to encounter: Lemma

Consider the LP problem:

(1)

$$\max \boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x}$$
$$\text{subject to } \boldsymbol{Ax} = \boldsymbol{b}$$
$$\boldsymbol{x} \geq \boldsymbol{0}$$

Then:

**(2)**

$$\boldsymbol{Ax} + \boldsymbol{z} = \boldsymbol{b} \ \ (if\ \boldsymbol{z}\ is\ the\ vector\ of\ artificial\ variables)$$
$$\therefore \boldsymbol{x}, \boldsymbol{z} \geq 0$$

Lemma:

The LP problem (1) is feasible if and only if the optimal value of the LP problem (2) is achieved if $\boldsymbol{z} = \boldsymbol{0}$ in the final step

*PROOF of Lemma:*

$\Longrightarrow$If $\boldsymbol{x}^\star$ is a feasible solution of (1), then $(\boldsymbol{x}^\star, \boldsymbol{0})$ is a feasible solution of the LP problem 2, and therefore is optimal. So $\boldsymbol{z} = \boldsymbol{0}$

$\Longleftarrow$ if, on the oter hand, the optimal value of minimising the $\sum_{i=1}^{n} z_i$ is zero; with solution of (2) being $(\boldsymbol{x}^\star, \boldsymbol{0})$ and thus $\boldsymbol{x}^\star$ is a feasible solution $A\boldsymbol{x}^\star = \boldsymbol{b}$ of (1).

## Example of non standard LP:

$$\max Z = 3x_1 + 5x_2$$

Subject to:

$$x_1 \leq 4$$

$$3x_1 + 2x_2 \geq 18$$
$$2x_2 \leq 12$$
$$with\ x_1, x_2 \geq 0$$

$$\therefore x_1 + x_3 = 4$$
$$3x_1 + 2x_2 - x_4 + \overline{x_6} = 18$$
$$2x_2 + x_5 = 12$$
$$with\ x_{1\to 5}; \bar{x}_6 \geq 0$$

$$\therefore first\ FCP: x_1 = x_2 = x_4 = 0; x_5 = 12; \bar{x}_6 = 18$$

## Two Phase Simplex algorithm:

We can find an initial basic feasible solution to the non standard LP problem:

$$\max Z = 3x_1 + 5x_2$$

Subject to:

$$x_1 \leq 4$$
$$3x_1 + 2x_2 \geq 18$$
$$2x_2 \leq 12$$
$$with\ x_1, x_2 \geq 0$$

By casting as the following:

$$\therefore x_1 + x_3 = 4$$
$$3x_1 + 2x_2 - x_4 + \overline{x_6} = 18$$
$$2x_2 + x_5 = 12$$
$$with\ x_{1\to 5}; \bar{x}_6 \geq 0$$

$$\therefore first\ FCP: x_1 = x_2 = x_4 = 0; x_5 = 12; \bar{x}_6 = 18$$

Eg: if

$$x_1 + x_2 = 2$$
$$2x_1 + x_2 \geq 1$$

We then have:

$$x_1 + x_2 - x_3 = 2$$
$$2x_1 + x_2 - x_4 + \bar{x}_5 = 1$$