# DISCRETE MATHS AND GRAPH THEORY: MATH2969 summary

## Contents

# Chapter 1: counting principles

1. Study of non-continuous maths
2. (eg, sequences, combinatorics, induction ect)

Eg: continuous function $y = x^2$; has analogous sequences $a_n = n^2$

$$\mathbb{N} = \{0,1,2 \dots\}$$

For finite set $X = \{x_1, x_2 \dots x_n\}, |X| = n$

The size (or cardinality) of $X$

## Theorem: bijecetion principle

If 2 sets $X$ and $Y$ are a bijection; $|X| = |Y|$

3. Bijection is 1-1 correspondence, and is a way of associating to each element of $X$ a corresponding element of $Y$

## Injective:

1-1: $f: X \rightarrow Y$ if $f(x_1) \neq f(x_2)\ \ for\ x_1 = x_2$

For each $y \in Y$, there is at most 1 $x \in X | f(x) = y$

## Surjective:

'onto': $f: X \rightarrow Y$, if each value of $Y$ is one of the values taken by the function (eg, if the function maps to all $\mathbb{N}$, then $Y = \mathbb{N}$

Bijective= surjective+injective

## Sum principle

If $X = A \cup B\ and\ A \cap B = \emptyset$

$$|X| = |A| + |B|$$

## Difference principle

For any $A \subset X$

$$|X \backslash A| = |X| - |A|$$

Preimage: $f^{-1}(y) \coloneqq \{x \in X | f(x) = y\}$

$$|X| = \sum_{y \in Y} |f^{-1}(y)|$$

### Rounding notation:

$$\lceil x \rceil \coloneqq x \text{ rounded up to nearest integer}$$
$$\lfloor x \rfloor \coloneqq x \text{ rounded down to nearest integer}$$

$$\lfloor x \rceil = x \text{ rounded (up or down)to nearest integer}$$

$$x - 1 < \lfloor x \rfloor \leq x; \quad x \leq \lceil x \rceil < x + 1$$

## Pigeonhole principle:

$$f: X \to Y, \text{there must be a } y \in Y|$$
$$|f^{-1}(y)| \geq \left\lceil \frac{|X|}{|Y|} \right\rceil$$

### Cartesian product:

$$X \times Y \coloneqq \{(x, y) | x \in X, y \in Y\}$$

### Product principle:

$$|A| \times |B| \times |C| \times \ldots = |A \times B \times C \ldots|$$
$$|X^n| = |X|^n$$

## Overcounting principle:

If $f: X \to Y, and\ |f^{-1}(y)| = m$

$$then\ |Y| = \frac{|X|}{m}$$

## Number of subsets of a set:

If $Y \subset X = \{1,2, \ldots n\}$

$$\therefore number\ of\ subses\ Y = 2^n$$

## Number of ordered selections $(y_1 \ldots y_k)$

### Injective functions

Is analogous to umber of injective functions

$$number\ of\ injective\ function's from\ X \to Y\ is$$

$$= \boldsymbol{n_k} = \frac{n!}{(n-k)!} \;\; ((-m)! = \infty) \quad (n)(n-1)(n-2) + (n-k+1)$$

## Number of unordered selections:

$$= \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

## Binomial theorem:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Proof: $(a+b)^n = (a+b)(a+b)\dots(a+b)\;\;(n\ times)$

$timesing\ everything\ together\ with\ the\ distributive\ law\ will\ get\ multiples\ of\ a^{n-k}b^{k}\ '$

Eg: $aababbb \dots ba\ ect$

$$number\ of\ ways\ to\ choose\ where\ b\ sits\ from\ n\ places\ is\ \binom{n}{k}$$

Properties:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\sum_{n=0}^{n} \binom{n}{k} = 2^n$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \begin{cases} 0 & n \geq 1 \\ 1 & n = 0 \end{cases}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = (\binom{2n}{n}$$

## Multinomial coefficients:

Let X be a set with $|X| = n$, and $n_1, n_2 \dots n_m \in \mathbb{N} | \sum n_i = n$

$$let\ \binom{n}{n_1, n_2, n_3 \dots n_m} = the\ number\ of\ ways\ of\ choosing\ subsets\ A_1 \dots A_m$$

From $X; |A_i| = n_i; X = A_1 \cup A_2 \dots \cup A_m \; (disjoint\ union)$

$$\binom{n}{n_1, n_2, n_3 \dots n_m} = \frac{n!}{n_1!\, n_2!\, n_3! \dots n_m!}$$

Proof: if X lies on a line $X = 1,2 \dots n$

$number\ of\ permuations\ of\ X = n!$

$if\ we\ were\ to\ choose\ the\ first\ n_1\ to\ be\ in\ A_1, the\ net\ n_2 to\ be\ in\ A_2 ect$

$then\ the\ number\ of\ ways\ to\ rearagne\ X\ ino\ subsets\ A_i\ (where\ the\ subset\ A_i\ then\ stays\ the\ same)$

$$= \frac{n!}{n_1!\, n_2! \dots n_m!}$$

Eg: if there were 12 balls, and we wanted to divide them into 4 jars, of 2,3,5 and 2 balls in each respectively. Number of ways this can be done is $\binom{12}{2,3,5,2}$

## Multinomial theorem:

$$(a_1 + a_2 + \cdots + a_m)^n = \sum_{n_1 + n_2 + \cdots + n_m = n} \binom{n}{n_1, n_2, n_3 \dots n_m} a_1^{n_1} a_2^{n_2} \dots a_n^{n_m}$$

Proof analogous to binomial theorem:

Multinomial and binomial link:

$$\binom{n}{n_1, n_2, n_3 \dots n_m} = \binom{n}{n_m} \binom{n - n_m}{n_{m-1}} \dots \binom{n_1}{n_1} = \prod_{i=1}^{m} \binom{\sum_{j=1}^{i} k_j}{k_i}$$

### Unordered selections with repetition

The number of unordered selections with repetition is:

$$\binom{n + k - 1}{k}$$

Proof: having chosen elements of each type in a row: eg type 1, type 2.. type n (where $t_1 + t_2 + \cdots + t_n = k$)

We can separate these groups of each type with $n - 1$ lines

Total objects= $n + k - 1$

Number of ways to choose placemet of each object is $\binom{n+k-1}{k}$

Eg: the number of ways to give 3 identical boxes to 5 people, and each person can get multiple boxes:

$$= \binom{3+5-1}{3} = \binom{7}{3}$$

## Inclusion/ exlusion principle:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Ect:

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum|A_i| - \sum|A_i \cap A_j| + \sum|A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1}\sum|A_1 \cap A_2 \cap \ldots \cap A_n|$$

$$\therefore |A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{k=1}^{n}\left[(-1)^{k-1} \sum_{i_1,i_2,i_3 \ldots i_n=1}^{n} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_n}|\right]$$

Proof: suppose $I$ is the subset $I = \{1,2,3 \ldots n\}, of\ each\ x \in A_i$

Then $A_{i_1} \cap A_{i_2} \cap \ldots A_{i_k}$ contains $x\ iff\ \ (i_1, i_2, i_3 \ldots i_k \in I)$. So, for each $k$, x is counted $\binom{|I|}{k}$

So the total number on the RHS = $\sum_{k=1}^{|I|}\binom{|I|}{k}(-1)^{k-1} = 1$

$$\therefore RHS = LHS$$

Eg: how many numbers from 1-1000 are divisible by 2,3 and 5 (note need to divide by lowest common multiple)

$$\# = \left\lfloor\frac{1000}{2}\right\rfloor + \left\lfloor\frac{1000}{3}\right\rfloor + \left\lfloor\frac{1000}{5}\right\rfloor - \left\lfloor\frac{1000}{6}\right\rfloor - \left\lfloor\frac{1000}{10}\right\rfloor - \left\lfloor\frac{1000}{15}\right\rfloor + \left\lfloor\frac{1000}{30}\right\rfloor$$

## Dearangements of a set:

Eg: there area n guys, with n hats what is the number of ways no one gets the correct hat?

(answer is integer closest to $\frac{n!}{e}$)

A derangement of $\{1,2, \dots n\}\ is\ a\ permutation\ of\ f: \{1 \dots n\} \to \{1 \dots n\}$

$$|\ f(x_i) \neq i\ \ \forall i$$

## Number of derangements:

$$\#\ derangements\ of\ a\ set\ = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}$$

Proof: the number of not derangements:

$$A_i = \{permutations\ of\ f | f(i) = i\}$$
$$\therefore\ set\ of\ not\ derangements\ is\ (A_1 \cup A_2 \cup \dots \cup A_n)$$

$$by \frac{inclusion}{exclusion}: \ |A_{i_1} \cup \dots \cup A_{i_k}| = (n-k)!$$
$$\therefore |A_1 \cup \dots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (n-k)!$$
$$\therefore number\ of\ derangements:$$
$$\sum_{k=0}^{n} (-1)^k \frac{n!}{k!}$$

(eg: how many ways are there to send 4 notes to 4 people, so that no one gets the letter intended for them?)

$$= \sum_{k=0}^{4} \frac{(-1)^k 4!}{k!} = \frac{4!}{0!} - \frac{4!}{1!} + \frac{4!}{2!} - \frac{4!}{3!} + \frac{4!}{4!}$$

## Number of surjective functions:

$$number\ of\ surjective\ functions\ from\ |X| = m, |Y| = k$$
$$\#\ surjective\ f: X \to Y$$
$$= \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^m$$

Proof: $A_i = \{f: X \to Y | f(x_i) = i\ \forall x \in X\}$

$$x = \{1..m\}; Y = \{1 \dots k\}$$
$$\therefore number\ of\ non\ surjective\ functions = |A_1 \cup A_2 \dots A_k|$$
$$= \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} (k-j)^m$$
$$\therefore number\ of\ surjecve\ functions = k^m - \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} (k-j)^m$$
$$= \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^m$$

Eg: how many ways is there to arrange 5 students to 3 teachers, so that each teacher has at least 1 student:

$$= \sum_{i=0}^{3} (-1)^i \binom{3}{i} (3-i)^5 = (3^5) - 3(2)^5 + 3(1)^5 - 0$$

### Stirling numbers

For $n, k \in \mathbb{N}$,

$$S(n,k)$$

Is the number of ways to write $X = \{1 \dots n\}$ as a disjoint union of $k$ non-empty subsets. The number of partitions of $\{1..n\}$ into $k$ blocks.

Note:

$$S(n.n) = 1$$
$$S(n,0) = 0 \quad (n \geq 1)$$
$$S(n,2) = 2^{n-1} - 1$$
$$S(n,n-1) = \binom{n}{2}$$

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

Proof: for $X = \{1 \dots n\}$ we can set $n$ by itself:

$\dots | \dots | \dots | \dots \dots \dots .. | n = S(n-1,k-1)$ *ways to partition* $n-1$ *into* $k-1$ *blocks*

Or: into a set in a set with other elements:

$\dots | \dots | \dots n \dots | \dots$

$= kS(n-1,k)$ (*partitioning* $n - 1$ *into* $k$ *blocks, and* $k$ *selections of where* n *could have gone*)

| | | k | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| | 1 | 1 | | | | | | |
| | 2 | 1 | 1 | | | | | |
| | 3 | 1 | 4 | 1 | | | | |
| m | 4 | 1 | 11 | 11 | 1 | | | |
| | 5 | 1 | 26 | 66 | 26 | 1 | | |
| | 6 | 1 | 57 | 302 | 302 | 57 | 1 | |
| | 7 | 1 | 120 | 1191 | 2416 | 1191 | 120 | 1 |

THeorum:

## Number of surjective functions

$$\begin{aligned}&if\ |X| = m; |Y| = k\\ &number\ of\ surjective\ functions\ f: X \to Y\\ \\ &= k!\, S(m,k)\end{aligned}$$

Proof:

$(blob\ diagrams)$: $the\ preimages\ of\ Y\ form\ a\ partition\ of\ X\ and\ are\ disjoint\ (to\ be\ a\ function)$
$by\ definition, number\ of\ ways\ for\ this\ partition\ is\ S(m,k)$
$then\ k!\ ways\ to\ arrange\ partitions$
$\therefore k!\, S(m,k)$

Eg: arrange 5 students into 3 tutors:

$$= 3!\, S(5,3)$$

## Formula for $n^m$

$$n^m = \sum_{k=0}^{m} k!\, S(m,k) \binom{n}{k}$$

Proof: $X = \{1..m\}; Y = \{1..n\};\ number\ of\ f: X \rightarrow = n^m$

$let\ |f\{1 \dots m\}| = k$

$number\ of\ functions\ with\ this\ will\ be\ k!\, S(m,k); and\ there\ are\ \binom{n}{k}\ wyas\ to\ choose\ these\ subsets$

$\therefore \# = \sum_{k=0}^{m} k!\, S(m,k) \binom{n}{k}$

$$S(m,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^m$$

# Chapter 2: recursion and induction

A sequence: is

$$a_n; where\ a_n is\ a\ function\ f: \mathbb{N} \rightarrow \mathbb{R}\ (or \mathbb{C})$$

## Recurrence relation:

$$a_n = g(a_1, a_2 \dots a_{n-1});\ \ with\ initial\ values\ a_0, a_1 \dots a_k$$

Eg: $a_n = a_{n-1} + d\ \ is\ arithmetic\ sequences$

$$a_n = r\, a_{n-1}\ \ is\ geometric\ sequence$$

Fibonaci: $F_n = F_{n-1} + F_{n-2}\,; F_0 = 0; F_1 = 1$

Catalan numbers: $C_n = c_0 c_{n-1} + c_1 c_{n-2} \cdots c_{n-1} c_0 = \sum_{m=0}^{n-1} c_m c_{n-m-1}$

(catalan numbers are the number of balanced strings of left and right brackets):

Proof:

$$\{ (\dots)( \quad ) . . ( \quad )\}\{(\dots .)\}$$
$$Left := 2(m+1) \ brackets, Right = 2(n-m-1) brackets$$
$$\therefore = sum \ of \ all \ these \ products$$

## Unravelling procedure:

$$eg: t_n = t_{n-1} + n \, ; t_0 = o$$
$$\therefore t_n = t_{n-2} + (n-1) + n = t_{n-3} + (n-2) + (n-1) + n$$
$$= \cdots = t_0 + 1 + 2 + \cdots + n$$
$$= \sum n = \frac{n}{2}(n+1)$$

Eg 2: $a_n = 3a_{n-1} + 1; \;\; a_0 = 1$

$$\therefore a_n = 3(3a_{n-2} + 1) + 1 = 3^2 a_{n-2} + 3 + 1$$
$$= 3^2(3a_{n-3} + 1) + 3 + 1$$
$$= 3^3 a_{n-3} + 3^2 + 3 + 1 = \cdots = 3^{n+1} a_0 + 3^n + \cdots + 1$$
$$= Geometric \ series: \; = \frac{3^n - 1}{2}$$

## Mathematical induction:

Suppose a statement $S(n) \;\; for \; n \in \mathbb{N}$:

1. Verify base case(s) $S(0), S(1)$ …
2. Assume for $S(n)$
3. Show it holds for $S(n+1) \;\; (using \ induction \ hypothesis)$

Eg: show $S(n): 1 + r + r^2 + \cdots r^n = \frac{r^{n+1}-1}{r-1}$

$$S(0): LHS = 1; RHS = \frac{r-1}{r-1} = 1 = LHS \ holds \ S(0)$$
$$assume \ S(n) true$$
$$S(n+1): 1 + r + \cdots + r^n + r^{n+1} = \frac{r^{n+1} - 1}{r - 1} + r^{n+1} \; \big(by \ induction \ hypthesis \ S(n)\big)$$

$$= \frac{(r^{n+1} - 1 + r^{n+2} - r^{n+1})}{r-1} = \frac{r^{n+2}-1}{r-1} \therefore holds\ S(n+1)$$

$\therefore proved\ by\ the\ prinicple\ of\ mathematical\ induction\ for\ n \geq 0$

$eg\ 2: find\ solution\ to:\ a_0 = 0; a_n = a_{n-1} + n^3\ (n \geq 1)$

$a_n = a_{n-2} + (n-1)^3 + n^3 = \cdots = \sum k^3$

| $n$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $a_n$ | 0 | 1 | 9 | 36 | 100 |

Conjecture: $S(n) = a_n = \left(\frac{n(n+1)}{2}\right)^2$

$S(0): LhS = RHS = 0$

$holds\ S(0)\ assume\ true\ for\ S(n)$

$\therefore for\ S(n+1):$

$$a_{n+1} = \sum k^3 + (n+1)^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3\ \ (by\ induction\ hypthesis)$$

$$= \frac{(n+1)^2}{4}(n^2+4n+4) = \frac{(n+1)^2}{4}(n+2)^2 = \left(\frac{(n+1)(n+2)}{2}\right)^2 \therefore holds\ S(n+1)$$

Eg 3: NOTE if we have

$a_n\ as\ a\ function\ of\ more\ than\ just\ one\ other\ a_{n-1}, we\ need\ more\ base\ cases$

Eg: fibonaci:

$$F_n = F_{n-1} + F_{n-2} < 2^n$$

$let\ S(n)\ denote\ statement\ F_n < 2^n$

$F_0 = 0; F_1 = 1$

$S(0): F_0 = 0 < 2^0 = 1$

$S(1): F_1 = 1 < 2^1$

$\therefore holds\ for\ n = 0,1$

$assume\ true\ for\ S(n)\ and\ S(n-1):$

$\therefore F_{n-1} < 2^{n-1}$

$F_n < 2^n$

$Considering\ S(n+1):$

$\therefore F_{n+1} = F_n + F_{n-1} < 2^{n-1} + 2^n\ \ (by\ induction\ hypothesis)$

$$= 2^{n+1}\left(\frac{1}{4}+\frac{1}{2}\right) = \frac{3}{4}(2^{n+1}) < 2^{n+1}$$
$\therefore S(n+1) holds$

Need for well defined base cases:

$S(n) = \text{any n people have the same height:}$
$S(1)\ is\ true\ (1\ person\ has\ the\ same\ height\ as\ themself)$
$assume\ S(n)$
$S(n$
$+1):\ can\ be\ broken\ into\ separate\ groups\ of\ n\ people, all\ of\ which\ n\ have\ the\ same\ height, so\ all$
$+1\ must\ have\ same\ height, \therefore\ everyone\ has\ equal\ height$
$PROBLEM: S(1)\ does\ not\ imple\ S(2):\ just\ because\ i\ am\ the\ same\ height\ as\ myself,$
$i\ am\ not\ the\ same\ height\ compared\ with\ o$
$another$

# Solving classes of recursion relations

## Homogeneous linear recursion relations:

The $kth$ or homogeneous linear recursion relation is of the form:

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_k a_{n-k} \quad (n \geq k \geq 1; r_k \neq 0)$$

Eg: $a_n = \lambda a_{n-1}\ \ 1st\ order$

$a_n = a_{n-2} + 6a_{n-4}\ \ (4th\ order)$

$c_n = c_0 c_{(n-1)}\ \ (not\ linear, so\ no\ order)$

## Characterstic polynomial:

$$p(x) = x^k - r_1 x^{k-1} - r_2 x^{k-2} - \cdots - r_k = 0 \quad (note: move\ everything\ to\ the\ left)$$

## Therorem:

$let\ \lambda\ be\ a\ root\ of\ p(x); then\ the\ sequence\ a_n = \lambda^n\ \ is\ a\ solution$
$if\ \lambda\ is\ a\ repeated\ root\ of\ multiplicity\ m; then\ the\ sequence$
$n\lambda^n;\ n^2\lambda^n \ldots n^{m-1}\lambda^n\ \ (or\ n^s\lambda^n\ \ for\ 1 \leq s \leq m-1)\ are\ also\ solutions$

Proof of 1:

$$\lambda^k = r_1\lambda^{k-1} + \cdots + r_k$$
$$\therefore \lambda^n = r_1\lambda^{k-1} + \cdots + r_k\lambda^{n-k}$$
$$\therefore a_n = \lambda^n \; is \; a \; solution$$

Proof of 2:

We want to show all sequences of $n^s\lambda^n$ are solutions

$$p(x) = x^k - r_r x^{k-1} - r_2 x^{k-2} - \cdots - r_k$$
$$\therefore \lambda^k - r_1\lambda^{k-1} - \cdots - r_k = 0$$
$$\therefore x^n - r_1 x^{n-1} - \cdots - r_k x^{n-k} = x^{n-k}p(x) \;\; has \; a \; factor \; of \; (x-\lambda)^{m-1}$$
$$\therefore nx^{n-1} - (n-1)r_1x^{n-2} - \cdots - (n-k)r_k x^{n-k-1} \;\; has \; a \; factor \; of \; (x-\lambda)^{m-2}$$
$$\therefore (\times x) \colon nx^n - (n-1)r_1 x^{n-1} - \cdots - r_k(n-k)x^{n-k} \;\; has \; factor \; (x-\lambda)^{m-2}$$
$$\therefore a_n = n\lambda^n \; is \; a \; solution$$
$$\therefore we \; keep \; on \; deriving \; and \; mutliplying \; by \; x \; to \; show \; that \; all \; sequences:$$
$$a_n = n^s\lambda^n \;\; are \; solutions$$

## Theroerm:

If $a_n$ and $b_n$ are solutions then

$$c_n = Aa_n + Bb_n \;\; is \; also \; a \; solution$$

Proof:

$$c_n = A(r_1a_{n-1} + r_2a_{n-2} + \cdots + r_ka_{n-k}) + B(r_1b_{n-1} + \cdots + r_kb_{n-k})$$
$$= r_1(Aa_{n-1} + Bb_{n-1}) + \cdots + r_k(Aa_{n-k} + Bb_{n-k})$$
$$= r_1c_{n-1} + \cdots + r_kc_{n-k}$$

# General solution to linear homogenous recurrence relation::

$$if \; p(x) = (x-\lambda_1)^{m_1}(x-\lambda_2)^{m_2} \ldots \left(x-\lambda_p\right)^{m_p}$$

Then general solution is linear combination of

$$n^s\lambda_1^{n-k}, n^s\lambda_2^{n-k} \ldots n^2\lambda_p^{n-k}$$

Proof: where multiplicyt of $\lambda_i \; is \; 1$

$$A_1\lambda_1^n + A_2\lambda_2^n + \cdots + A_k\lambda_k^n \; satisfies:$$

$$n = 0{:}\ A_1 + A_2 + \cdots A_k = a_0$$
$$n = 1{:} A_1\lambda_1 + \cdots + A_k\lambda_k = a_1$$
$$\ldots$$
$$n = k-1{:} A_1\lambda_1^{k-1} + \cdots + A_k\lambda_k^{k-1} = a_{n-1}$$
$$which\ has\ unique\ solution\ if{:}$$
$$\det\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \ldots & \lambda_k \\ \ldots & & & & \\ \lambda_1^{k-1} & \ldots & & & \lambda_k^{k-1} \end{pmatrix} \neq 0$$
$$\therefore \prod_{1\le i,j\le k} (a_j - a_i) \neq 0$$
$$which\ holds$$

Eg:

Solve $a_n = 7a_{n-1} - 12a_{n-2};\quad a_0 = 2; a_1 = 5$

$$\therefore p(x) = x^2 - 7x - 12 = (x-4)(x-3) = 0$$
$$x = 3{,}4$$
$$\therefore a_n = A3^n + B4^n$$
$$a_0 = A + B = 2$$
$$a_1 = 3A + 4B = 5$$
$$\Longrightarrow A = 3; B = -1$$
$$\therefore a_n = 3^{n+1} - 4^n$$

(note: always check with your solution with the initial conditions after to check)

Eg2:

$$a_n = 3a_{n-2} - 2a_{n-3}$$

$$a_0 = 0; a_1 = 1; a_2 = 2\ \ (note{:} 3rd\ order, need\ cubic)$$
$$\therefore x^3 - 3x - 2 = 0$$
$$(x-1)^2(x+2) = 0$$
$$\therefore x = 1; -2$$
$$\therefore a_n = A(1)^n + Bn(1)^n + C(-2)^n = A + Bn + C(-2)^n$$
$$a_0{:} A + B + C = 0$$
$$a_1{:} A + B - 2C = 1$$
$$a_2{:} A + 2B + 4C = 2$$
$$this\ solves\ out\ to\ be{:} A = C = 0; B = 1$$
$$\therefore a_n = n$$

## Non homogenous linear recurrence relations

$$kth\ or\ non\ homogenours\ linear\ recurrence\ relation\ is\ of\ the\ form$$