

DISCRETE MATHS AND GRAPH THEORY: MATH2969 summary

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Chapter 1: counting principles

1. Study of non-continuous maths
2. (eg, sequences, combinatorics, induction ect)

Eg: continuous function $y = x^2$; has analogous sequences $a_n = n^2$

$$\mathbb{N} = \{0, 1, 2, \dots\}$$

For finite set $X = \{x_1, x_2, \dots, x_n\}$, $|X| = n$

The size (or cardinality) of X

Theorem: bijection principle

If 2 sets X and Y are a bijection; $|X| = |Y|$

3. Bijection is 1-1 correspondence, and is a way of associating to each element of X a corresponding element of Y

Injective:

1-1: $f: X \rightarrow Y$ if $f(x_1) \neq f(x_2)$ for $x_1 \neq x_2$

For each $y \in Y$, there is at most 1 $x \in X$ | $f(x) = y$

Surjective:

'onto': $f: X \rightarrow Y$, if each value of Y is one of the values taken by the function (eg, if the function maps to all \mathbb{N} , then $Y = \mathbb{N}$)

Bijjective= surjective+injective

Sum principle

If $X = A \cup B$ and $A \cap B = \emptyset$

$$|X| = |A| + |B|$$

Difference principle

For any $A \subset X$

$$|X \setminus A| = |X| - |A|$$

Preimage: $f^{-1}(y) := \{x \in X | f(x) = y\}$

$$|X| = \sum_{y \in Y} |f^{-1}(y)|$$

Rounding notation:

$\lceil x \rceil := x$ rounded up to nearest integer

$\lfloor x \rfloor := x$ rounded down to nearest integer

$\lceil x \rceil = x$ rounded (up or down) to nearest integer

$$x - 1 < \lceil x \rceil \leq x; \quad x \leq \lfloor x \rfloor < x + 1$$

Pigeonhole principle:

$f: X \rightarrow Y$, there must be a $y \in Y$ |

$$|f^{-1}(y)| \geq \left\lceil \frac{|X|}{|Y|} \right\rceil$$

Cartesian product:

$$X \times Y := \{(x, y) | x \in X, y \in Y\}$$

Product principle:

$$|A| \times |B| \times |C| \times \dots = |A \times B \times C \dots|$$
$$|X^n| = |X|^n$$

Overcounting principle:

If $f: X \rightarrow Y$, and $|f^{-1}(y)| = m$

$$\text{then } |Y| = \frac{|X|}{m}$$

Number of subsets of a set:

If $Y \subset X = \{1, 2, \dots, n\}$

$$\therefore \text{number of subsets } Y = 2^n$$

Number of ordered selections $(y_1 \dots y_k)$

Injective functions

Is analogous to number of injective functions

number of injective function's from $X \rightarrow Y$ is

$$= n_k = \frac{n!}{(n-k)!} \quad ((-m)! = \infty)$$

Number of unordered selections:

$$= \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Binomial theorem:

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$$

Proof: $(a+b)^n = (a+b)(a+b) \dots (a+b)$ (n times)

timesing everything together with the distributive law will get multiples of $a^{n-k}b^k$,

Eg: aababbb ... ba ect

number of ways to choose where b sits from n places is $\binom{n}{k}$

Properties:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\sum_{k=0}^n \binom{n}{k} = 2^n$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} = \begin{cases} 0 & n \geq 1 \\ 1 & n = 0 \end{cases}$$

$$\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$$

Multinomial coefficients:

Let X be a set with $|X| = n$, and $n_1, n_2, \dots, n_m \in \mathbb{N} \mid \sum n_i = n$

let $\binom{n}{n_1, n_2, n_3, \dots, n_m} =$ the number of ways of choosing subsets $A_1 \dots A_m$

From $X; |A_i| = n_i; X = A_1 \cup A_2 \dots \cup A_m$ (disjoint union)

$$\binom{n}{n_1, n_2, n_3 \dots n_m} = \frac{n!}{n_1! n_2! n_3! \dots n_m!}$$

Proof: if X lies on a line $X = 1, 2 \dots n$

number of permutations of $X = n!$

if we were to choose the first n_1 to be in A_1 , the next n_2 to be in A_2 etc

then the number of ways to rearrange X into subsets A_i (where the subset A_i then stays the same)

$$= \frac{n!}{n_1! n_2! \dots n_m!}$$

Eg: if there were 12 balls, and we wanted to divide them into 4 jars, of 2, 3, 5 and 2 balls in each respectively. Number of ways this can be done is $\binom{12}{2, 3, 5, 2}$

Multinomial theorem:

$$(a_1 + a_2 + \dots + a_m)^n = \sum_{n_1 + n_2 + \dots + n_m = n} \binom{n}{n_1, n_2, n_3 \dots n_m} a_1^{n_1} a_2^{n_2} \dots a_m^{n_m}$$

Proof analogous to binomial theorem:

Multinomial and binomial link:

$$\binom{n}{n_1, n_2, n_3 \dots n_m} = \binom{n}{n_m} \binom{n - n_m}{n_{m-1}} \dots \binom{n_1}{n_1} = \prod_{i=1}^m \binom{i}{k_i}$$

Unordered selections with repetition

The number of unordered selections with repetition is:

$$\binom{n + k - 1}{k}$$

Proof: having chosen elements of each type in a row: eg type 1, type 2.. type n (where $t_1 + t_2 + \dots + t_n = k$)

We can separate these groups of each type with $n - 1$ lines

Total objects = $n + k - 1$

Number of ways to choose placement of each object is $\binom{n + k - 1}{k}$

Eg: the number of ways to give 3 identical boxes to 5 people, and each person can get multiple boxes:

$$= \binom{3 + 5 - 1}{3} = \binom{7}{3}$$

Inclusion/exclusion principle:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Etc:

$$\begin{aligned} |A_1 \cup A_2 \cup \dots \cup A_n| &= \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| - \dots \\ &+ (-1)^{n-1} \sum |A_1 \cap A_2 \cap \dots \cap A_n| \end{aligned}$$

$$\therefore |A_1 \cup A_2 \cup \dots \cup A_n| = \sum_{k=1}^n \left[(-1)^{k-1} \sum_{i_1, i_2, i_3, \dots, i_n=1}^n |A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_n}| \right]$$

Proof: suppose I is the subset $I = \{1, 2, 3 \dots n\}$, of each $x \in A_i$

Then $A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}$ contains x iff $(i_1, i_2, i_3 \dots i_k \in I)$. So, for each k , x is counted $\binom{|I|}{k}$

So the total number on the RHS = $\sum_{k=1}^{|I|} \binom{|I|}{k} (-1)^{k-1} = 1$
 $\therefore RHS = LHS$

Eg: how many numbers from 1-1000 are divisible by 2, 3 and 5 (note need to divide by lowest common multiple)

$$\# = \left\lfloor \frac{1000}{2} \right\rfloor + \left\lfloor \frac{1000}{3} \right\rfloor + \left\lfloor \frac{1000}{5} \right\rfloor - \left\lfloor \frac{1000}{6} \right\rfloor - \left\lfloor \frac{1000}{10} \right\rfloor - \left\lfloor \frac{1000}{15} \right\rfloor + \left\lfloor \frac{1000}{30} \right\rfloor$$

Dearrangements of a set:

Eg: there are n guys, with n hats what is the number of ways no one gets the correct hat?

(answer is integer closest to $\frac{n!}{e}$)

A derangement of $\{1, 2, \dots, n\}$ is a permutation of $f: \{1 \dots n\} \rightarrow \{1 \dots n\}$
 $| f(x_i) \neq i \forall i$

Number of derangements:

$$\# \text{ derangements of a set} = \sum_{k=0}^n (-1)^k \frac{n!}{k!}$$

Proof: the number of not derangements:

$A_i = \{\text{permutations of } f \mid f(i) = i\}$
 $\therefore \text{set of not derangements is } (A_1 \cup A_2 \cup \dots \cup A_n)$

by $\frac{\text{inclusion}}{\text{exclusion}}$: $|A_{i_1} \cup \dots \cup A_{i_k}| = (n - k)!$

$$\therefore |A_1 \cup \dots \cup A_n| = \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} (n - k)!$$

$\therefore \text{number of derangements:}$

$$\sum_{k=0}^n (-1)^k \frac{n!}{k!}$$

(eg: how many ways are there to send 4 notes to 4 people, so that no one gets the letter intended for them?)

$$= \sum_{k=0}^4 \frac{(-1)^k 4!}{k!} = \frac{4!}{0!} - \frac{4!}{1!} + \frac{4!}{2!} - \frac{4!}{3!} + \frac{4!}{4!}$$

Number of surjective functions:

number of surjective functions from $|X| = m, |Y| = k$

surjective $f: X \rightarrow Y$

$$= \sum_{i=0}^k (-1)^i \binom{k}{i} (k-i)^m$$

Proof: $A_i = \{f: X \rightarrow Y \mid f(x_i) = i \forall x \in X\}$

$x = \{1..m\}; Y = \{1..k\}$

\therefore number of non surjective functions = $|A_1 \cup A_2 \dots A_k|$

$$= \sum_{j=1}^k (-1)^{j-1} \binom{k}{j} (k-j)^m$$

\therefore number of surjective functions = $k^m - \sum_{j=1}^k (-1)^{j-1} \binom{k}{j} (k-j)^m$

$$= \sum_{j=0}^k (-1)^j \binom{k}{j} (k-j)^m$$

Eg: how many ways is there to arrange 5 students to 3 teachers, so that each teacher has at least 1 student:

$$= \sum_{i=0}^3 (-1)^i \binom{3}{i} (3-i)^5 = (3^5) - 3(2^5) + 3(1)^5 - 0$$

Stirling numbers

For $n, k \in \mathbb{N}$,

$$S(n, k)$$

Is the number of ways to write $X = \{1..n\}$ as a disjoint union of k non-empty subsets. The number of partitions of $\{1..n\}$ into k blocks.

Note:

$$S(n, n) = 1$$

$$S(n, 0) = 0 \quad (n \geq 1)$$

$$S(n, 2) = 2^{n-1} - 1$$

$$S(n, n-1) = \binom{n}{2}$$

$$S(n, k) = S(n - 1, k - 1) + kS(n - 1, k)$$

Proof: for $X = \{1 \dots n\}$ we can set n by itself:

$\dots | \dots | \dots | \dots \dots \dots | n = S(n - 1, k - 1)$ ways to partition $n - 1$ into $k - 1$ blocks

Or: into a set in a set with other elements:

$\dots | \dots | \dots n \dots | \dots$
 $= kS(n - 1, k)$ (partitioning $n - 1$ into k blocks, and k selections of where n could have gone)

		k						
	1	2	3	4	5	6	7	
1	1							
2	1	1						
3	1	4	1					
m	4	1	11	11	1			
5	1	26	66	26	1			
6	1	57	302	302	57	1		
7	1	120	1191	2416	1191	120	1	

Theorem:

Number of surjective functions

if $|X| = m; |Y| = k$

number of surjective functions $f: X \rightarrow Y$

$$= k! S(m, k)$$

Proof:

(blob diagrams): the preimages of Y form a partition of X and are disjoint (to be a function) by definition, number of ways for this partition is $S(m, k)$

then $k!$ ways to arrange partitions

$$\therefore k! S(m, k)$$

Eg: arrange 5 students into 3 tutors:

$$= 3! S(5,3)$$

Formula for n^m

$$n^m = \sum_{k=0}^m k! S(m, k) \binom{n}{k}$$

Proof: $X = \{1..m\}; Y = \{1..n\}$; number of $f: X \rightarrow Y = n^m$

let $|f^{-1}\{1..m\}| = k$

number of functions with this will be $k! S(m, k)$; and there are $\binom{n}{k}$ ways to choose these subsets

$$\therefore \# = \sum_{k=0}^m k! S(m, k) \binom{n}{k}$$

$$S(m, k) = \frac{1}{k!} \sum_{j=0}^k (-1)^j \binom{k}{j} (k-j)^m$$

Chapter 2: recursion and induction

A sequence: is

$$a_n; \text{ where } a_n \text{ is a function } f: \mathbb{N} \rightarrow \mathbb{R} \text{ (or } \mathbb{C})$$

Recurrence relation:

$$a_n = g(a_1, a_2 \dots a_{n-1}); \text{ with initial values } a_0, a_1 \dots a_k$$

Eg: $a_n = a_{n-1} + d$ is arithmetic sequences

$$a_n = r a_{n-1} \text{ is geometric sequence}$$

Fibonacci: $F_n = F_{n-1} + F_{n-2}; F_0 = 0; F_1 = 1$

Catalan numbers: $C_n = c_0 c_{n-1} + c_1 c_{n-2} + \dots + c_{n-1} c_0 = \sum_{m=0}^{n-1} c_m c_{n-m-1}$

(catalan numbers are the number of balanced strings of left and right brackets):

Proof:

$\{ (\dots) (\dots) \dots (\dots) \{ (\dots) \}$
 $Left := 2(m+1) \text{ brackets}, Right = 2(n-m-1) \text{ brackets}$
 $\therefore := \text{sum of all these products}$

Unravelling procedure:

$$\begin{aligned} \text{eg: } t_n &= t_{n-1} + n; t_0 = 0 \\ \therefore t_n &= t_{n-2} + (n-1) + n = t_{n-3} + (n-2) + (n-1) + n \\ &= \dots = t_0 + 1 + 2 + \dots + n \\ &= \sum_{k=1}^n k = \frac{n}{2}(n+1) \end{aligned}$$

Eg 2: $a_n = 3a_{n-1} + 1; a_0 = 1$

$$\begin{aligned} \therefore a_n &= 3(3a_{n-2} + 1) + 1 = 3^2 a_{n-2} + 3 + 1 \\ &= 3^2(3a_{n-3} + 1) + 3 + 1 \\ &= 3^3 a_{n-3} + 3^2 + 3 + 1 = \dots = 3^{n+1} a_0 + 3^n + \dots + 1 \\ &= \text{Geometric series: } = \frac{3^{n+1} - 1}{2} \end{aligned}$$

Mathematical induction:

Suppose a statement $S(n)$ for $n \in \mathbb{N}$:

1. Verify base case(s) $S(0), S(1) \dots$
2. Assume for $S(n)$
3. Show it holds for $S(n+1)$ (using induction hypothesis)

Eg: show $S(n): 1 + r + r^2 + \dots + r^n = \frac{r^{n+1} - 1}{r - 1}$

$$S(0): LHS = 1; RHS = \frac{r - 1}{r - 1} = 1 = LHS \text{ holds } S(0)$$

assume $S(n)$ true

$$S(n+1): 1 + r + \dots + r^n + r^{n+1} = \frac{r^{n+1} - 1}{r - 1} + r^{n+1} \text{ (by induction hypothesis } S(n))$$

$$= \frac{(r^{n+1} - 1 + r^{n+2} - r^{n+1})}{r - 1} = \frac{r^{n+2} - 1}{r - 1} \therefore \text{holds } S(n + 1)$$

\therefore proved by the principle of mathematical induction for $n \geq 0$

eg 2: find solution to : $a_0 = 0; a_n = a_{n-1} + n^3 (n \geq 1)$

$$a_n = a_{n-2} + (n - 1)^3 + n^3 = \dots = \sum k^3$$

n	0	1	2	3	4
a_n	0	1	9	36	100

Conjecture: $S(n) = a_n = \left(\frac{n(n+1)}{2}\right)^2$

$$S(0): LHS = RHS = 0$$

holds $S(0)$ assume true for $S(n)$

\therefore for $S(n + 1)$:

$$a_{n+1} = \sum k^3 + (n + 1)^3 = \left(\frac{n(n + 1)}{2}\right)^2 + (n + 1)^3 \text{ (by induction hypothesis)}$$

$$= \frac{(n + 1)^2}{4} (n^2 + 4n + 4) = \frac{(n + 1)^2}{4} (n + 2)^2 = \left(\frac{(n + 1)(n + 2)}{2}\right)^2 \therefore \text{holds } S(n + 1)$$

Eg 3: NOTE if we have

a_n as a function of more than just one other a_{n-1} , we need more base cases

Eg: fibonacci:

$$F_n = F_{n-1} + F_{n-2} < 2^n$$

let $S(n)$ denote statement $F_n < 2^n$

$$F_0 = 0; F_1 = 1$$

$$S(0): F_0 = 0 < 2^0 = 1$$

$$S(1): F_1 = 1 < 2^1$$

\therefore holds for $n = 0, 1$

assume true for $S(n)$ and $S(n - 1)$:

$$\therefore F_{n-1} < 2^{n-1}$$

$$F_n < 2^n$$

Considering $S(n + 1)$:

$$\therefore F_{n+1} = F_n + F_{n-1} < 2^{n-1} + 2^n \text{ (by induction hypothesis)}$$

$$= 2^{n+1} \left(\frac{1}{4} + \frac{1}{2} \right) = \frac{3}{4} (2^{n+1}) < 2^{n+1}$$

$\therefore S(n+1)$ holds

Need for well defined base cases:

$S(n)$ = any n people have the same height:

$S(1)$ is true (1 person has the same height as themselves)

assume $S(n)$

$S(n$

+ 1): can be broken into separate groups of n people, all of which n have the same height, so all

+ 1 must have same height, \therefore everyone has equal height

PROBLEM: $S(1)$ does not imply $S(2)$: just because i am the same height as myself,

i am not the same height compared with o

another

Solving classes of recursion relations

Homogeneous linear recursion relations:

The k th or homogeneous linear recursion relation is of the form:

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \dots + r_k a_{n-k} \quad (n \geq k \geq 1; r_k \neq 0)$$

Eg: $a_n = \lambda a_{n-1}$ 1st order

$$a_n = a_{n-2} + 6a_{n-4} \quad (4\text{th order})$$

$$c_n = c_0 c_{(n-1)} \quad (\text{not linear, so no order})$$

Characteristic polynomial:

$$p(x) = x^k - r_1 x^{k-1} - r_2 x^{k-2} - \dots - r_k = 0 \quad (\text{note: move everything to the left})$$

Theorem:

let λ be a root of $p(x)$; then the sequence $a_n = \lambda^n$ is a solution

if λ is a repeated root of multiplicity m ; then the sequence

$n\lambda^n; n^2\lambda^n \dots n^{m-1}\lambda^n$ (or $n^s\lambda^n$ for $1 \leq s \leq m-1$) are also solutions

Proof of 1:

$$\begin{aligned}\lambda^k &= r_1\lambda^{k-1} + \dots + r_k \\ \therefore \lambda^n &= r_1\lambda^{n-1} + \dots + r_k\lambda^{n-k} \\ \therefore a_n &= \lambda^n \text{ is a solution}\end{aligned}$$

Proof of 2:

We want to show all sequences of $n^s\lambda^n$ are solutions

$$\begin{aligned}p(x) &= x^k - r_1x^{k-1} - r_2x^{k-2} - \dots - r_k \\ \therefore \lambda^k - r_1\lambda^{k-1} - \dots - r_k &= 0 \\ \therefore x^n - r_1x^{n-1} - \dots - r_kx^{n-k} &= x^{n-k}p(x) \text{ has a factor of } (x - \lambda)^{m-1} \\ \therefore nx^{n-1} - (n-1)r_1x^{n-2} - \dots - (n-k)r_kx^{n-k-1} &\text{ has a factor of } (x - \lambda)^{m-2} \\ \therefore (\times x): nx^n - (n-1)r_1x^{n-1} - \dots - r_k(n-k)x^{n-k} &\text{ has factor } (x - \lambda)^{m-2} \\ \therefore a_n = n\lambda^n &\text{ is a solution} \\ \therefore \text{we keep on deriving and multiplying by } x &\text{ to show that all sequences:} \\ a_n = n^s\lambda^n &\text{ are solutions}\end{aligned}$$

Theroerm:

If a_n and b_n are solutions then

$$c_n = Aa_n + Bb_n \text{ is also a solution}$$

Proof:

$$\begin{aligned}c_n &= A(r_1a_{n-1} + r_2a_{n-2} + \dots + r_ka_{n-k}) + B(r_1b_{n-1} + \dots + r_kb_{n-k}) \\ &= r_1(Aa_{n-1} + Bb_{n-1}) + \dots + r_k(Aa_{n-k} + Bb_{n-k}) \\ &= r_1c_{n-1} + \dots + r_kc_{n-k}\end{aligned}$$

General solution to linear homogenous recurrence relation::

$$\text{if } p(x) = (x - \lambda_1)^{m_1}(x - \lambda_2)^{m_2} \dots (x - \lambda_p)^{m_p}$$

Then general solution is linear combination of

$$n^s\lambda_1^{n-k}, n^s\lambda_2^{n-k} \dots n^2\lambda_p^{n-k}$$

Proof: where multiplicity of λ_i is 1

$$A_1\lambda_1^n + A_2\lambda_2^n + \dots + A_k\lambda_k^n \text{ satisfies :}$$

$$\begin{aligned}
n = 0: & A_1 + A_2 + \dots + A_k = a_0 \\
n = 1: & A_1\lambda_1 + \dots + A_k\lambda_k = a_1 \\
& \dots \\
n = k - 1: & A_1\lambda_1^{k-1} + \dots + A_k\lambda_k^{k-1} = a_{n-1}
\end{aligned}$$

which has unique solution if:

$$\det \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \dots & \lambda_k \\ \dots & \dots & \dots & \dots & \dots \\ \lambda_1^{k-1} & \dots & \dots & \dots & \lambda_k^{k-1} \end{pmatrix} \neq 0$$

$$\therefore \prod_{1 \leq i, j \leq k} (a_j - a_i) \neq 0$$

which holds

Eg:

$$\text{Solve } a_n = 7a_{n-1} - 12a_{n-2}; \quad a_0 = 2; a_1 = 5$$

$$\therefore p(x) = x^2 - 7x + 12 = (x - 4)(x - 3) = 0$$

$$x = 3, 4$$

$$\therefore a_n = A3^n + B4^n$$

$$a_0 = A + B = 2$$

$$a_1 = 3A + 4B = 5$$

$$\Rightarrow A = 3; B = -1$$

$$\therefore a_n = 3^{n+1} - 4^n$$

(note: always check with your solution with the initial conditions after to check)

Eg2:

$$a_n = 3a_{n-2} - 2a_{n-3}$$

$$a_0 = 0; a_1 = 1; a_2 = 2 \quad (\text{note: 3rd order, need cubic})$$

$$\therefore x^3 - 3x - 2 = 0$$

$$(x - 1)^2(x + 2) = 0$$

$$\therefore x = 1; -2$$

$$\therefore a_n = A(1)^n + Bn(1)^n + C(-2)^n = A + Bn + C(-2)^n$$

$$a_0: A + B + C = 0$$

$$a_1: A + B - 2C = 1$$

$$a_2: A + 2B + 4C = 2$$

$$\text{this solves out to be: } A = C = 0; B = 1$$

$$\therefore a_n = n$$

Non homogenous linear recurrence relations

*k*th or non homogenous linear recurrence relation is of the form