

Binomial Model

Stock Price Dynamics

- The value of an option at maturity depends on the price of the underlying stock at maturity.
- The value of the option today depends on the expected value of the option at maturity and hence on the expected stock price at maturity.
- Objective : To specify a mathematical model for the movements (dynamics) in the price of the underlying stock over time

Discrete Random Variables	<p>A discrete random variable is a variable X which is to take on one value by chance from a set of possible values</p> <ul style="list-style-type: none"> • The realisation of a random variable represents the value of X that occurs (X is no longer random)
Probability and Distribution Functions	<p>The probabilities associated with a discrete random variable X are given by the probability function :</p> $\text{Prob}(X = x_i) = p(x_i)$ <p>Distribution function :</p> $\text{Prob}(X \leq x_i) = \sum_{x \leq x_i} p(x_1) + p(x_2) + \dots + p(x_i)$

- Expectations and Moments

⇒ The expected value of discrete random variable is

$$E[X] = \sum_{i=1}^n x_i p(x_i)$$

⇒ If X is a random variable and g is a function, then $g(X)$ is also a random variable and

$$E[g(X)] = \sum_{i=1}^n g(x_i) p(x_i)$$

⇒ Mean of X is $E(X)$ and variance of X is $\text{Var}[X] = E[(X - E[X])^2]$

- Stochastic Processes

⇒ A stochastic process is a sequence of related random variable over time i.e there is a physical link between them

⇒ Let X be rv measured over time. Then sequence of rv's $X_0, X_1, X_2, \dots, X_T$ is a discrete time stochastic process

⇒ The sample path of stochastic process(sp) represents the sequence of particular values of each X that occurs i.e process is no longer random

⇒ The sequence $S_0, S_1, S_2, \dots, S_T$ is a discrete time sp for the stock price over time

⇒ To price options we need a model for the movements in underlying stock over time

⇒ Model movements in stock prices over time by a stochastic process

⇒ Number of different stochastic processes may be used

The Key Idea – Riskless Hedge

- The riskless hedge is the key concept underlying most of the standard option pricing models
 - ⇒ Including binomial model
 - ⇒ Based on the assumption on no arbitrage
- Riskless hedge refers the fact that a combination of options and stock in appropriate proportions can be used to replicate the payoff on a risk free bond
- Irrelevance of the Stock Expected Return
 - ⇒ Same option price when probability upward movement is 0.5 or 0.9. It is natural to assume that as the probability of an upward movement as stock price increase value of call increase and value of put decreases
 - ⇒ We are not valuing the option in absolute terms but calculating its value in terms of the price of the underlying stock.

- ⇒ Probabilities of future up and down are already incorporated into the price of the stock so we do not need to take them into account again when valuing the option in term of stock price
- Implication of the riskless hedge approach for option valuation are :
 - 1) Option is risky but it is not necessary to determine the risk premium applicable to the option in order to value it
 - ⇒ To value a risky asset we would normally need to know the expected rate of return on the asset
 - ⇒ Since the payoff on the portfolio of stock and option is risk free, the return on the portfolio is risk free rate
 - ⇒ Work out the value of the option given current stock price
 - ⇒ Stock is risky + option is risky but Stock and Option together is riskless
 - 2) In accordance with the non arbitrage assumption this is a relative valuation approach – we value the option relative to price of the stock but we don't require the stock to be fairly priced
 - 3) If the observed market price of the option differs from the NA price – arbitrage opportunity
 - 4) We did not need to know the probability that stock price will rise or fall
 - 5) The riskless hedge is the basis for black scholes and binomial model

One Period Model

- 1) European call option on a stock with a strike K, maturity T years
- 2) The current price of the stock is S and over each period of length years, the stock price either increases by a factor u with probability p or decrease by a factor d with probability 1 – q where u,d,q are constant
- 3) The stock pay no dividends over the life of the option
- 4) Frictionless market
- 5) The nominal risk free rate is constant at r% p.a where $u > e^{rt} > d$
- 6) No arbitrage
- 7) One time period to maturity.

Value of Option at maturity

- If stock price rises : $f_u = \max [uS - K]$
- If stock price falls : $f_d = \max [dS - K]$

Value of the portfolio :

Up movement	$S_0 u \Delta - f_u$	Two are equal when : $S_0 u \Delta - f_u = S_0 d \Delta - f_d$
Down Movement	$S_0 d \Delta - f_d$	

$$\text{So } \Delta = \frac{f_u - f_d}{S_0 u - S_0 d}$$

Value of the portfolio at the end is same irrespective of whether stock price rises or falls. The value of the portfolio is therefore riskless. (not affected by the stock price risk(stock irrelevance))

- Cost of setting up the portfolio is $S_0 u \Delta - f$
- Price of the value of the option is

$$f = \frac{p f_u + (1 - p) f_d}{e^{rT}} \text{ where } p = \frac{e^{rT} - d}{u - d}$$

- p is the probability of an upward movement/probability of option price being worth f_u/f_d not probability of the stock price going up
- Value of the option does not depend on the expected return of the stock (depends on the volatility expectation)
- f is a relative pricing relationship. For a given S, this is the correct f

- Δ is the delta of the option and specifies the number of shares to be bought for each option sold at the start of the period
- In practice, determining u and d from stock volatility
 $u = e^{\sigma\sqrt{\Delta t}}$ where else $d = 1/u$

Two Period Model

- 1) European call option on a stock with a strike K , maturity T years
- 2) The current price of the stock is S and over each period of length years, the stock price either increases by a factor u with probability p or decrease by a factor d with probability $1 - p$ where u, d, p are constant
- 3) The stock pay no dividends over the life of the option
- 4) Frictionless market
- 5) The nominal risk free rate is constant at $r\%$ p.a where $u > e^{rt} > d$
- 6) No arbitrage
- 7) Two period of equal length maturity

Current value of the option

$$f = \frac{p^2 f_{uu} + 2p(1-p)f_{ud} + (1-p)^2 f_{dd}}{e^{2rT}}$$

- Since u and d are the same for each period, the tree recombines. Non recombine trees explode as the number of periods increase
- The risk neutral hedge needs to be re-balanced at the start of each period if the hedge is to be maintained over the life of the option.
- With put option, recalculate the f_u and f_d
- Delta (Δ) of a stock option is the ratio of the change in the price of the stock option to the change in the price of the underlying stock. It is the number of units of the stock that we should hold for each option shorted in order to create riskless portfolio. This is known as delta hedging.
- Delta of a call option is positive where as delta of put option is negative.
- Purpose of valuing an option
 - a) The expected return from all traded securities is the risk free interest rate
 - b) Future cash flows can be valued by discounting their expected values at the risk free interest rate

Three Period Model

$$f = \frac{p^3 f_{ddd} + 3p^2(1-p)f_{uud} + 3p(1-p)^2 f_{ddu} + (1-p)^3 f_{ddd}}{e^{3r\Delta t}}$$

Risk Neutral Valuation

- 1) Risk neutral world – investor do not care about risk in valuing assets. They do not increase the expected return they require from an investment to compensate for increased risk.
- 2) The discount rate used for the expected payoff on an option is the risk free rate.
- 3) In risk neutral world, the expected return on the stock is the risk free rate
- 4) Risk neutral world assumes risk neutrality only for the purpose of valuing the option even, though most investor are risk averse
- 5) It can be shown that value of the option derived in risk neutral world is the same as the value of the option in a risk averse world

The current value of an option is equal to the present value of its expected future payoff in a risk neutral world using the risk free rate as the discount rate

$$f = e^{-rT} E * [f_T]$$

In the real world, it is not easy to know the correct discount rate to apply to the expected payoff but with risk neutral it is convenient because we know that in a risk neutral world, the expected return on all assets is the risk free rate.

The n period model

Current value of the option :

$$f = \frac{\sum_{j=0}^n \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} f_{n,j}}{e^{nr\Delta t}}$$

American Option

Procedure is to work back through the tree from the end to the beginning, testing at each node to see whether early exercise is optimal. The value of option at the final nodes is the same as for European option. At earlier nodes,

1. Value is given by equation
2. Payoff if exercise early. If payoff is more than value of the option, then exercise the option

Option on other assets

Continuous Dividend Yield	$p = \frac{e^{(r-q)T} - d}{u - d}$ <p>Stock paying a known dividend yield at rate q. The total return from dividend and capital gains is r. So r - q.</p>
Option on Stock Indices	Assume underlying index provide a dividend yield rate q. So similar assumption as dividend yield
Currencies	A foreign currency regarded as an asset providing a yield at the foreign risk free rate of interest. In this case $a = e^{(r-r_f)\Delta t}$
Futures	Cost nothing to take a long or a short position in a future contract. It follows that in a risk neutral world, a future price should have an expected growth rate of zero. In here : $p = \frac{1-d}{u-d}$ <p>since future cost nothing.</p>

Hedging Strategies

What happen if the market price of an option differs from its theoretical price/value?

- The one period binomial model is based on the 1 period riskless hedge portfolio

$$\pi = \Delta S - f^{theory}$$

So long bond = long Δ Share + short 2 option
 - If we observed market price of the option differs from the theoretical value of the option f^{theory} then an arbitrage opportunity exists
 - The riskless hedge portfolio shows us how to lock in an arbitrage profit.
- 1) The positions of the hedge portfolio follows from the fact that you can hedge a short option with an equivalent long option
 - ⇒ Since long shares and short bonds is equivalent to long option, it can be used to hedge an equivalent short option
 - 2) Sometimes hedging strategy calls for a reverse hedge. Here the above absolute positions are reversed but the relative positions are the same
 - ⇒ Hedge a long option with a short stock and long bond

- 3) The 1 period hedge states that an appropriate combination of option and stock can be used to create a riskless portfolio
 \Rightarrow This can be extended to that an appropriate combination of any two of the securities (options, stock and bonds) can be used to create a portfolio equal to the third.
- 4) In the n period model, delta changes from period to period. So riskless hedge portfolio need to be rebalance at the start of each period
- 5) The $\Delta \pi$ are node dependent, the delta hedging strategy is path dependent
- 6) This give a perfect hedge in theory
- 7) If hedge is maintained by keeping the number of options constant and buying and selling bonds, then minimum profit is the initial mispricing of the option. But if hedge is maintained by keeping the number of shares constant and buying/selling options and bond a loss may result.
- 8) If theoretical price of option is equal to the market price of the option, the portfolio can be used to hedge the option and is called a hedging strategy
- 9) If theoretical price is not equal to market price, the hedge portfolio can be used to lock in an arbitrage profit and is called an arbitrage strategy.

Practical Limitation

- a) Sometimes delta might be 0.267 and number like this cause rounding error to hedge the appropriate number
- b) How often to rebalance? In theory, continuously replace and rebalance. However, in reality we have to take into account the transaction cost.