MAST10005 Calculus 1 – Detailed Notes

About these notes

These notes cover all seven topics of the University of Melbourne MAST10005 Calculus 1 lecture notes: set theory, complex numbers, functions, vectors, differential calculus, integral calculus and differential equations.

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Topic 1: Set theory

Topic 1 sets up the language of sets ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, intervals, subsets, unions, intersections, complements, Cartesian products) and the rules for manipulating inequalities, which later give domains, ranges and the intervals on which a function is increasing or decreasing.

1.0 Real numbers

Set	Description	Examples and facts
$\mathbb{N}$ (natural numbers)	The positive integers, or counting numbers: {1, 2, 3, 4, ...}	Any natural number is 1, 2, 3, 4 and so on
$\mathbb{Z}$ (integers)	The whole numbers: positive, negative and zero	..., −2, −1, 0, 1, 2, 3, ... (shown on a number line)
$\mathbb{Q}$ (rational numbers)	Numbers that can be written as fractions m/n, where m and n are integers [with $n \neq 0$]	1/2, 78/23, 0/5. There are infinitely many rationals, each can be written in infinitely many ways, and every integer is a rational number
$\mathbb{R}$ (real numbers)	The irrational numbers joined together with the rational numbers	Irrational numbers cannot be written as fractions, e.g. π, e, $\sqrt{2}$, $\log_2(5)$

- There are infinitely many irrational numbers. In a sense there are "more" of them than there are rationals.
- The sets are nested: $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$ (used as the "obvious example of subsets" in 1.14).
- To convince ourselves that irrational numbers actually exist, one needs a **proof by contradiction**, shown below.

Proof by contradiction that $\sqrt{2}$ is irrational

1. Assume $\sqrt{2} \in \mathbb{Q}$. Then $\sqrt{2} = a/b$ with $a, b \in \mathbb{Z}$, $b \neq 0$, and a and b have no common divisors.
2. Squaring gives $2 = a^2/b^2$, so $2b^2 = a^2$.
3. So a^2 is a multiple of 2, which means a must be even. Let $a = 2c$, where $c \in \mathbb{Z}$.
4. Then $2b^2 = (2c)^2 = 4c^2$, so $b^2 = 2c^2$.
5. This implies b is also even, which contradicts the assumption that a and b have no common divisors. Hence $\sqrt{2} \notin \mathbb{Q}$, i.e. $\sqrt{2}$ is irrational.

1.13 Bounded intervals

For real numbers a, b with a equal to or less than b:

Notation	Meaning	Descriptive form
(a, b)	Interval of real numbers between a and b, exclusive (open circles at both ends)	$\{x \in \mathbb{R} \mid a < x < b\}$
[a, b]	All real numbers from a to b, inclusive (filled dots at both ends)	$\{x \in \mathbb{R} \mid a \leq x \leq b\}$
$(-\infty, a)$	Real numbers **strictly** less than a	$\{x \in \mathbb{R} \mid x < a\}$
(a, ∞)	Real numbers **strictly** greater than a	$\{x \in \mathbb{R} \mid x > a\}$

[Clarification: a round bracket leaves the endpoint out and a square bracket includes it]

List-of-elements form, infinite lists and the empty set

- A set with only finitely many elements (a "small set") can be written in **list of elements** form:

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\{x \in \mathbb{R} \mid x^2 - 1 = 0\} = \{-1, 1\}, \quad \{x \in \mathbb{Z} \mid x^2 - 1 < 0\} = \{0\}, \quad \{x \in \mathbb{Z} \mid x^3 - x = 0\} = \{-1, 0, 1\}
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- **Example.** The set of prime numbers less than 30: A = {2, 3, 5, 7, 11, 13, 17, 19, 23, 29}.
- Some infinite sets can be written with dots, which often makes the set easier to visualise: $\{(2k + 1)\pi \mid k \in \mathbb{Z}\} = \{\ldots, -3\pi, -\pi, \pi, 3\pi, \ldots\}$.
- There is usually only one set with no elements: the **empty set**, written ø. It can also be written {}, so ø = {}. The notation is useful when expressing solutions of equations and inequalities: $\{x \in \mathbb{R} \mid x^2 + 1 = 0\} = ø$ and $\{x \in \mathbb{R} \mid \cos(x) > 1\} = ø$.

1.14 Subsets

- $A \subseteq B$ means every element of A is also an element of B. Read "A is a subset of B", "A is contained in B" or "B contains A". In symbols: $x \in A \Rightarrow x \in B$.
- To **prove** $A \subseteq B$: assume x is an element of A and use valid mathematical reasoning to conclude that x is an element of B.
- To **prove A is not a subset of B** ($A \not\subset B$): find an x in A that is not in B.
- The empty set ø is a subset of every set.
- **Worked example 1.** Prove $A \subseteq B$ where $A = \{4n \mid n \in \mathbb{Z}\}$ and $B = \{2m + 2 \mid m \in \mathbb{Z}\}$. In list form A = {..., 0, 4, 8, 12, 16, ...} and B = {..., −2, 0, 2, 4, 6, 8, 10, ...}.
  - Let $x \in A$. Then $x = 4n$ for some $n \in \mathbb{Z}$.
  - We need $4n = 2m + 2$, which gives $m = 2n - 1$. Since $n \in \mathbb{Z}$, $m = 2n - 1 \in \mathbb{Z}$.
  - Check: $2m + 2 = 2(2n - 1) + 2 = 4n = x$, so $x \in B$. Hence $A \subseteq B$.
- **Worked example 2.** Prove $A \not\subset B$ where $A = \{3n + 1 \mid n \in \mathbb{Z}\}$ and $B = \{6m + 1 \mid m \in \mathbb{Z}\}$. In list form A = {..., −2, 1, 4, 7, 10, ...} and B = {..., −5, 1, 7, 13, 19, ...}.
  - $4 \in A$ because $(3 \times 1) + 1 = 4$.
  - Suppose $4 \in B$. Then $4 = 6m + 1$ for some $m \in \mathbb{Z}$, so $6m = 3$ and $m = 1/2$. But 1/2 is not an integer, so $4 \notin B$. [the point is that $m = 1/2$ is not an integer.]. Therefore $A \not\subset B$.

1.15 Unions, intersections and set complements

Operation	Notation	Descriptive definition	In words
Union	$A \cup B$	$\{x \mid x \in A \text{ or } x \in B\}$	Elements that are in at least one of A or B
Intersection	$A \cap B$	$\{x \mid x \in A \text{ and } x \in B\}$	Elements that are in both A and B
Relative complement (difference)	$A \setminus B$	$\{x \in A \mid x \notin B\}$	Elements that are in A but not in B

Union examples

1. $A = \{x \in \mathbb{R} \mid x^2 > 1\}$ as a union of intervals: $x \in (-\infty, -1) \cup (1, \infty)$.
2. $A = \{x \in (-2\pi, 2\pi] \mid \sin(x) \leq 0\}$ as a union of intervals: $x \in [-\pi, 0] \cup [\pi, 2\pi]$.
3. $(2, 8) \cup [3, 10] = (2, 10]$.

Intersection examples

1. $(2, 8) \cap [3, 10] = [3, 8)$.
2. $(0, \sqrt{2}] \cap [\pi/2, 3) = \emptyset$ (because $\sqrt{2} \approx 1.41$ is less than $\pi/2 \approx 1.57$, the intervals do not overlap).

Complement examples

1. Find $\mathbb{R} \setminus \mathbb{Z}$ and $\mathbb{Z} \setminus \mathbb{R}$. $\mathbb{R} \setminus \mathbb{Z} = \ldots \cup (-1, 0) \cup (0, 1) \cup (1, 2) \cup \ldots$, while $\mathbb{Z} \setminus \mathbb{R} = \{x \in \mathbb{Z} \mid x \notin \mathbb{R}\} = \emptyset$. So $\mathbb{R} \setminus \mathbb{Z} \neq \mathbb{Z} \setminus \mathbb{R}$: it is **not** generally true that $A \setminus B = B \setminus A$.
2. Write the set of real numbers for which $f(x) = (x^5 - 2x + 4)/(x^2 - 1)$ makes sense, as a set complement and as a union of intervals. The expression fails where the denominator is zero ($x^2 - 1 = 0$):

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A = \mathbb{R} \setminus \{x \in \mathbb{R} \mid x^2 - 1 = 0\} = \mathbb{R} \setminus \{-1, 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)
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1.16 Cartesian products

- The **Cartesian product** of sets A and B is the set of ordered pairs with first component in A and second in B:

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A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}
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- **Example.** $A = \{0, 1, 5\}$, $B = \{e, \pi\}$: $A \times B = \{(0, e), (0, \pi), (1, e), (1, \pi), (5, e), (5, \pi)\}$.
- The best-known example is the plane, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. A typical point $(x, y) \in \mathbb{R}^2$ has horizontal coordinate x and vertical coordinate y.

3.02 Functional arithmetic

For two functions f and g (on the same domain A) we can define new functions:

Function	Definition
$f + g : A \to \mathbb{R}$	$(f + g)(a) = f(a) + g(a)$
$f - g : A \to \mathbb{R}$	$(f - g)(a) = f(a) - g(a)$
$fg : A \to \mathbb{R}$	$(fg)(a) = f(a)g(a)$
$f/g : A \to \mathbb{R}$	$(f/g)(a) = f(a)/g(a)$, which needs 0 not to be in the range of g
1/g (the **reciprocal** of g)	$(1/g)(a) = 1/g(a)$

- Example: **tan** is defined as the ratio of sine and cosine: $\tan : \mathbb{R} \setminus \{(2k + 1)\pi/2 \mid k \in \mathbb{Z}\} \to \mathbb{R}$, $\tan(x) = \sin(x)/\cos(x)$. We remove the points where $\cos(x) = 0$ (all odd multiples of $\pi/2$). The zeros of tan are precisely the zeros of sin (all multiples of π). The graph of tan has vertical asymptotes at $x = \pm\pi/2$, passes through $(\pi/4, 1)$ and $(-\pi/4, -1)$, and repeats every π.

Image and range

- The **image** of a subset $S \subseteq A$ under f is $f(S) = \{b \in B \mid b = f(s) \text{ for some } s \in S\}$.
- The image of the whole domain, $f(A) = \{b \in B \mid b = f(a) \text{ for some } a \in A\}$, is the **range** of f. It is the smallest valid codomain for the rule f.
- **Example.** $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 4x + 3 = (x - 1)(x - 3)$.
  - (a) $f([3, 5])$: $f(3) = 0$ and $f(5) = (5 - 1)(5 - 3) = 8$, so $f([3, 5]) = [0, 8]$.
  - (b) $f([1, 5]) = [-1, 8]$ (the vertex at $x = 2$ gives the minimum $f(2) = -1$).
  - (c) The range of f is $[-1, \infty)$.

3.03 Reciprocal trigonometric functions

Function	Definition	Graph features
cosec(x)	1/sin(x)	Vertical asymptotes at multiples of π; upper branches touch 1 at $\pi/2$ and lower branches touch -1 at $3\pi/2$
cot(x)	1/tan(x)	Decreasing between vertical asymptotes at multiples of π; zero at $\pi/2$ and $3\pi/2$
sec(x)	1/cos(x)	Vertical asymptotes at odd multiples of $\pi/2$; branches touch 1 at 0 and -1 at π

3.04 Types of functions

Type	Definition	Picture / meaning
Injective	$f(a_1) = f(a_2)$ automatically implies $a_1 = a_2$. Equivalently, $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$	f keeps the elements of A separated: no two distinct elements of A are given the same value
Surjective	$f(A) = B$, i.e. B is the range of f	For every $b \in B$ there exists at least one $a \in A$ with $f(a) = b$
Bijective	Both injective and surjective	f gives a precise correspondence between the elements of A and of B

- If A and B are subsets of $\mathbb{R}$, every increasing or decreasing function is injective. Proof for increasing f: suppose $a_1 \neq a_2$. If $a_1 > a_2$ then $f(a_1) > f(a_2)$ because f is increasing, in particular $f(a_1) \neq f(a_2)$; if $a_2 > a_1$ then $f(a_2) > f(a_1)$, again $f(a_1) \neq f(a_2)$.
- **Example 1.** $\log : (0, \infty) \to \mathbb{R}$ is increasing, therefore injective.
- **Example 2.** $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is not injective (it fails the **horizontal line test**). The largest intervals on which f is injective are $(-\infty, 0]$ and $[0, \infty)$ (the function stops being injective if the interval crosses the vertex).
- **Example 3.** All the surjective functions $\{1, 2, 3\} \to \{a, b\}$: there are six, listing (f(1), f(2), f(3)) as (a, a, b), (a, b, a), (a, b, b), (b, a, a), (b, a, b), (b, b, a).

Composition and inverses

Composition

- **Composition** takes the values of one function as the input of a second: $(g \circ f)(x) = g(f(x))$. Pictorially, $f : A \to B$, then $g : B \to C$, so $g \circ f : A \to C$.
- Two functions are equal when all the data attached to them are identical (same domain, codomain and rule; "the domains and ranges are equal").
- Properties of composition:
    a. If $f : A \to B$ then $f \circ id_A = id_B \circ f = f$.
    b. If $f : A \to B$, $g : B \to C$ and $h : C \to D$ then $h \circ (g \circ f) = (h \circ g) \circ f$ (associative).
- Is $f \circ g = g \circ f$? For the equality to make sense we need $A = B = C$, but even then composition is **not commutative**.

Asymptotes

- An **asymptote** of f is a straight line that the graph approaches either as x approaches a certain value or as x approaches infinity. They give **vertical**, **horizontal** or **oblique** (slant) asymptotes. Combining asymptotic behaviour with extrema and concavity helps us understand graphs.
- **Vertical asymptotes** arise when a denominator is zero, but a zero denominator does not guarantee one, and they may also arise without quotients ($y = \ln(x)$ has a vertical asymptote at $x = 0$).
- **Horizontal asymptotes** are associated with limiting behaviour as x approaches infinity (e.g. exponential functions). **Oblique asymptotes** are slant lines that the graph approaches, often for rational functions. To find an oblique or horizontal asymptote of a rational function, apply **polynomial long division**; this also helps find stationary and inflection points.
- It is incorrect to think of an asymptote as "a line that may not be crossed": that is true only of vertical asymptotes. A function may cross a horizontal or oblique asymptote many times.
- **Example 1.** Asymptotes of $f(x) = (x^2 + 2x + 1)/(x^2 - 1) = (x + 1)/(x - 1)$: $x - 1 = 0$, so there is a vertical asymptote at $x = 1$. [Added: the factor $x + 1$ cancels, so $x = -1$ is a removable hole rather than an asymptote, and $y = 1$ is a horizontal asymptote]
- **Example 2.** Asymptotes of $f(x) = (2x^3 - 3x^2 + 2x - 2)/(x^2 + 1)$. Polynomial long division gives $f(x) = 2x - 3 + 1/(x^2 + 1)$. So there are no vertical asymptotes, but there is an oblique asymptote $y = 2x - 3$.

Full curve-sketching example: $f(x) = x^2/(x + 1)$

Find the domain, asymptotes, intercepts, stationary points, increasing/decreasing intervals, local extrema, concavity and inflection points, then sketch.

Item	Result
1. Domain	$\mathbb{R} \setminus \{-1\}$
2. Asymptotes	Vertical asymptote at $x = -1$. Polynomial division gives $f(x) = x - 1 + 1/(x + 1)$, so there is an oblique asymptote $y = x - 1$
3. x and y intercepts	Both are 0 (the origin)
4. Stationary points	$f'(x) = 1 - 1/(x + 1)^2 = 0$ at $(0, 0)$ and $(-2, -4)$
5. Increasing	$(-\infty, -2) \cup (0, \infty)$
6. Decreasing	$(-2, 0)$ [strictly $(-2, -1) \cup (-1, 0)$, since f is undefined at -1]
7. Local extrema	Local maximum at $(-2, -4)$; local minimum at $(0, 0)$
8. Concave up	$f''(x) = 2/(x + 1)^3 > 0$ on $(-1, \infty)$
9. Concave down	$(-\infty, -1)$
10. Points of inflection	None (concavity changes only at $x = -1$, which is not in the domain)