MID-SEMESTER

Study Guide

Production Model | Solow Growth Model | Golden Rule | Romer Model

What this edition contains

Clear explanatory notes, formulas, original diagrams, worked examples, model comparisons and revision summaries. Dedicated quiz, practice-exam and mock-question sections have been removed.

Prepared as an independent educational revision resource. It is not an official university publication, past examination, assignment solution or marking guide.

Preview only

This sample contains representative notes and the topic list. It does not contain quiz or mock-exam questions.

Topics covered

A clear list for the StudentVIP listing and preview.

1	Mathematical toolkit: levels, changes, growth rates, powers, Cobb-Douglas algebra, marginal products and constant returns to scale.
2	Production Model: output, capital, labour, total factor productivity, diminishing marginal products and competitive factor markets.
3	Factor income shares, per-worker form, cross-country income differences, TFP decomposition and labour-force participation.
4	Solow Growth Model: saving, investment, consumption, capital accumulation and steady-state equilibrium.
5	Closed-form steady states, comparative statics, saving-rate shocks, transition dynamics and conditional convergence.
6	Golden Rule capital and consumption, including MPK = depreciation and the Cobb-Douglas saving-rate result.
7	Romer Model: research allocation, idea production, balanced growth, non-rival ideas and the level-growth trade-off.
8	Model comparison, graph interpretation, common exam traps, short-answer structure and final formula summaries.

Scope note

This edition is notes-only. Quiz questions, mock papers and answer-key sections are not included.

PART 1

Mathematical toolkit

1.1 Levels, changes and growth rates

These three objects are different:

FORMULA

$$\begin{aligned} \text{Level: } & X_t \\ \text{Change: } & \Delta X_{t+1} = X_{t+1} - X_t \\ \text{Growth rate: } & g_{X,t} = \frac{\Delta X_{t+1}}{X_t} = \frac{X_{t+1} - X_t}{X_t} \end{aligned}$$

If g_X is constant, then $X_t = X_0(1 + g_X)^t$.

INTUITION

A variable can have a **higher level** but the **same growth rate**. This is central to Solow: a higher saving rate raises the long-run levels of capital and output, but once the new steady state is reached, their growth rates return to zero in the core model.

1.2 Powers and Cobb–Douglas algebra

Useful rules:

$$x^a x^b = x^{a+b}, \qquad \frac{x^a}{x^b} = x^{a-b}, \qquad (x^a)^b = x^{ab}, \qquad x^{-a} = \frac{1}{x^a}.$$

If $k^{1-\alpha} = q$, raise both sides to $1/(1-\alpha)$:

$$k = q^{1/(1-\alpha)}.$$

WORKED EXAMPLE

Solve $0.24k^{1/3} = 0.06k$.

$$0.24k^{1/3} = 0.06k \Rightarrow 4 = k^{2/3} \Rightarrow k^* = 4^{3/2} = 8.$$

The exponent used at the final step is the reciprocal of $2/3$, which is $3/2$.

1.3 Marginal products

For $Y = AK^\alpha L^{1-\alpha}$:

FORMULA

$$MPK = \frac{\partial Y}{\partial K} = \alpha A K^{\alpha-1} L^{1-\alpha} = \alpha \frac{Y}{K},$$
$$MPL = \frac{\partial Y}{\partial L} = (1-\alpha) A K^\alpha L^{-\alpha} = (1-\alpha)\frac{Y}{L}.$$

Because $0 < \alpha < 1$, the exponent on K in MPK is negative and the exponent on L in MPL is negative. Holding the other input fixed, each marginal product falls as more of its own input is used.

2.3 Competitive factor markets

Firms take the wage w and rental rate r as given and maximise:

$$\Pi = F(K, L) - rK - wL.$$

The first-order conditions are:

FORMULA

$$r = MPK, \qquad w = MPL.$$

Firms hire an input until its extra output equals its real cost.

For Cobb–Douglas:

$$r = \alpha \frac{Y}{K}, \qquad w = (1-\alpha)\frac{Y}{L}.$$

Therefore:

FORMULA

$$\frac{rK}{Y} = \alpha, \qquad \frac{wL}{Y} = 1 - \alpha.$$

The exponents are also the factor income shares.

WORKED EXAMPLE

Let $Y = K^{1/3}L^{2/3}$ with $K = L = 8$ and $A = 1$.

$$Y = 8^{1/3}8^{2/3} = 8, \quad MPK = \frac{1}{3}\frac{8}{8} = \frac{1}{3}, \quad MPL = \frac{2}{3}\frac{8}{8} = \frac{2}{3}.$$

Capital income is $rK = (1/3)(8) = 8/3$ and labour income is $wL = (2/3)(8) = 16/3$. Together they exhaust output.

2.3.1 Elasticities in Cobb–Douglas production

The exponent on an input measures the percentage response of output to a one-percent change in that input, holding the other input fixed. For example, if $\alpha = 0.35$, a 10% increase in capital raises output by approximately 3.5% for a small change. The exact response is:

$$\frac{Y_{new}}{Y_{old}} = \left(\frac{K_{new}}{K_{old}}\right)^{0.35}.$$

This provides a quick way to distinguish a proportional change in an input from its smaller output effect under diminishing marginal productivity.

2.4 Per-worker form

Under CRS, divide the aggregate production function by L:

FORMULA

$$y \equiv \frac{Y}{L} = A\left(\frac{K}{L}\right)^{\alpha} = Ak^{\alpha}.$$

Living standards depend on productivity and capital per worker, not population size by itself.

INTUITION

Doubling both capital and labour doubles output, so output per worker is unchanged. Population can raise aggregate GDP without raising GDP per worker.

3.4 Steady-state equilibrium

A steady state is a level k^* that reproduces itself:

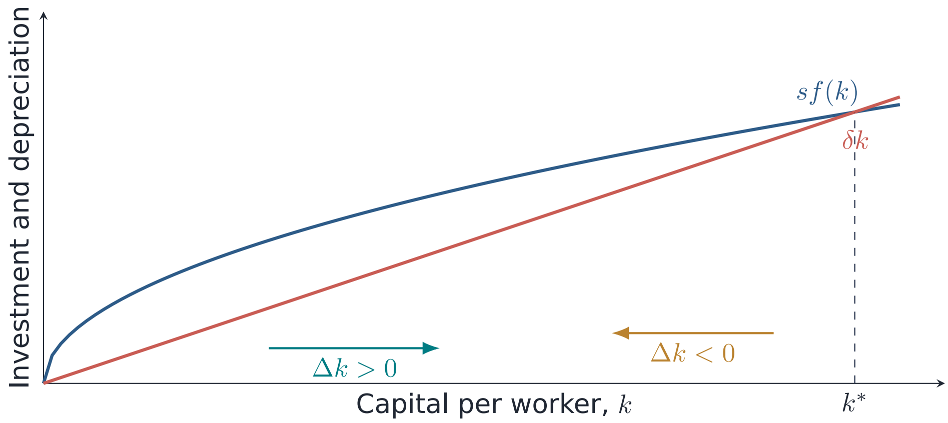
$$k_{t+1} = k_t = k^*, \qquad \Delta k = 0.$$

Therefore:

FORMULA

$$sf(k^*) = \delta k^*.$$

At the steady state, investment exactly replaces depreciated capital.



3.5 Closed-form steady state

For $f(k) = Ak^\alpha$:

$$sA(k^*)^\alpha = \delta k^*.$$

For a positive steady state, divide by $(k^*)^\alpha$:

FORMULA

$$k^* = \left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}}.$$

Then:

FORMULA

$$\begin{aligned} y^* &= A(k^*)^\alpha = A^{\frac{1}{1-\alpha}}\left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}}, \\ i^* &= sy^* = \delta k^*, \\ c^* &= (1-s)y^* = y^* - \delta k^*. \end{aligned}$$

WORKED EXAMPLE

Let $A = 1$, $\alpha = 1/3$, $s = 0.24$ and $\delta = 0.06$.

$$k^* = \left(\frac{0.24}{0.06}\right)^{3/2} = 4^{3/2} = 8,$$

$$y^* = 8^{1/3} = 2, \qquad i^* = 0.24(2) = 0.48, \qquad c^* = 1.52.$$

Check: $\delta k^* = 0.06(8) = 0.48 = i^*$.

PART 4

The Golden Rule

4.1 The objective

The ordinary steady state tells us where the economy settles for a given saving rate. The Golden Rule asks which steady state maximises long-run consumption per worker.

At any steady state, investment equals depreciation, so:

FORMULA

$$c^* = y^* - i^* = f(k^*) - \delta k^*.$$

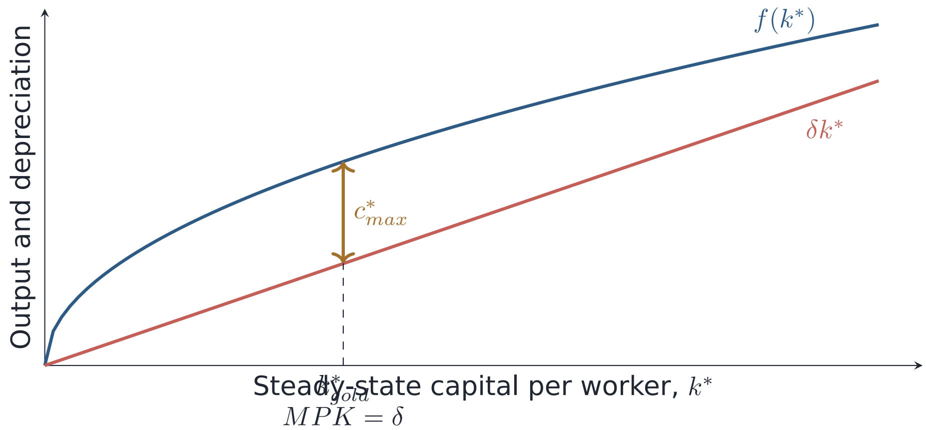
The slope of steady-state consumption with respect to capital is:

$$\frac{dc^*}{dk^*} = f'(k^*) - \delta.$$

The maximum therefore satisfies:

FORMULA

$$MPK = f'(k^*_{gold}) = \delta.$$



4.2 Cobb–Douglas Golden Rule

For $f(k) = Ak^\alpha$:

$$MPK = \alpha A k^{\alpha-1} = \delta.$$

Thus:

$$k^*_{gold} = \left(\frac{\alpha A}{\delta}\right)^{1/(1-\alpha)}.$$

Compare this with the ordinary steady state:

$$k^* = \left(\frac{sA}{\delta}\right)^{1/(1-\alpha)}.$$

Therefore:

5.3 Allocation of workers

Population L is fixed. A constant fraction ℓ works in research:

FORMULA

$$L_{at} = \ell L, \qquad L_{yt} = (1-\ell)L.$$

Researchers create ideas; production workers create goods and services.

The goods sector in the simplified framework used here is:

FORMULA

$$Y_t = A_t L_{yt} = A_t(1-\ell)L, \qquad y_t = \frac{Y_t}{L} = A_t(1-\ell).$$

5.4 Production of ideas

Assume idea growth is proportional to the number of researchers:

FORMULA

$$\frac{\Delta A_{t+1}}{A_t} = zL_{at} = z\ell L \equiv g.$$

z is research productivity and g is the constant growth rate of knowledge.

Therefore:

FORMULA

$$A_{t+1} = (1+g)A_t, \qquad A_t = A_0(1+g)^t.$$

Combining the knowledge path with goods production gives:

FORMULA

$$y_t = A_0(1-\ell)(1+g)^t, \qquad g = z\ell L.$$

WORKED EXAMPLE

Let $A_0 = 100$, $L = 50$, $\ell = 0.10$ and $z = 0.002$. Then:

$$g = z\ell L = 0.002(0.10)(50) = 0.01 = 1\%.$$

$$A_5 = 100(1.01)^5 = 105.10, \qquad A_{10} = 100(1.01)^{10} = 110.46.$$

The stock of ideas grows geometrically, not by adding one fixed number each period.

5.5 Balanced growth

In this simplified ideas model, $g = z\ell L$ is constant, so knowledge and output per person grow at a constant rate. This is a **balanced growth path**.

Solow	Transition dynamics: growth slows as the economy approaches a fixed steady state.
Romer	Balanced growth: endogenous variables grow at constant rates; there is no Solow-style approach to a fixed y^*.