33130 Mathematics 1

Complete Course Notes

FREE PREVIEW — Chapter 1 of 10

This preview includes

- Chapter 1: Vectors and Planes — in full

The complete set also covers

- Matrices
- Differentiation
- Logarithms
- Hyperbolic Functions
- Inverse Functions
- Integration
- Complex Numbers
- Differential Equations
- Power Series

Concise theory, key formulas, and fully worked examples for every topic.

1 Vectors and Planes

A **vector** is an object with magnitude and direction. It can be denoted using a letter with an arrow on top, e.g. $\vec{a}$, and expressed in terms of its components using:

- Parentheses: $\vec{a} = (a_1, a_2, a_3)$
- Angled brackets: $\vec{a} = \langle a_1, a_2, a_3 \rangle$
- Unit vector notation: $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$

For a 2D coordinate system where $(x, y) \in \mathbb{R}^2$, the vector components can be found using:

- $\vec{a_x} = |\vec{a}| \cos\theta$
- $\vec{a_y} = |\vec{a}| \sin\theta$

Magnitude: $|\vec{a}| = \sqrt{\sum_{i=1}^{n} a_i^2}$

1.1 Dot (Scalar) Product

The dot product of two vectors results in a **scalar** quantity.

$\vec{a} \cdot \vec{b} = \sum_{i=1}^{n} a_i b_i$

The angle between $\vec{a}$ and $\vec{b}$ is given by $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$.

When calculating the angle, the two vectors must be placed so their tails are at the same location.

1.2 Unit Vectors

A unit vector has magnitude exactly 1 and represents pure direction.

- Denoted with a hat, e.g. $\hat{\imath} = (1, 0, 0)$
- The unit vector in the direction of $\vec{a}$ is $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$

1.3 Vector Projection

A vector projection is the "shadow" of one vector cast onto another — the component of the first vector lying in the direction of the second.

- $\vec{a_1} = \text{proj}_{\vec{b}}\vec{a}$ is the projection of $\vec{a}$ onto $\vec{b}$.

Scalar projection of $\vec{a}$ onto $\vec{b}$: $\text{comp}_{\vec{b}}\vec{a} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

Vector projection of $\vec{a}$ onto $\vec{b}$: $\text{proj}_{\vec{b}}\vec{a} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\vec{b}$

1.4 Cross (Vector) Product

The cross product of two vectors results in a **normal vector**, perpendicular to both.

$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$.

To find the component form, use the determinant method with unit vectors in the first row:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = (a_y b_z - a_z b_y)\hat{\imath} - (a_x b_z - a_z b_x)\hat{\jmath} + (a_x b_y - a_y b_x)\hat{k}$$

Multiply each unit vector by the determinant of its "minor" matrix (the 2×2 matrix left after removing the vector's row and column). The direction is given by the **right-hand rule**:

- Align your fingers in the direction of the first vector.
- Bend those fingers toward the second vector.
- The thumb points in the direction of the cross product.

Properties of dot and cross products

- Cross product is anti-commutative: $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$. Dot product is commutative.
- Cross product is *not* associative: $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$. Dot product cannot be associative (three vectors don't give a scalar).
- Cross product is distributive: $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$. Dot product is also distributive.
- Scalar multiplication: $(k\vec{a}) \times \vec{b} = k(\vec{a} \times \vec{b})$; $(k\vec{a}) \cdot \vec{b} = k(\vec{a} \cdot \vec{b})$.
- $\vec{a} \times \vec{a} = \vec{0}$ since $\sin(0) = 0$. Meanwhile $\vec{a} \cdot \vec{a} = |\vec{a}|^2$.
- $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$, since $\vec{a}$ is perpendicular to $\vec{a} \times \vec{b}$.

Area of a parallelogram with adjacent sides $\vec{a}, \vec{b}$: $|\vec{a} \times \vec{b}|$
Area of a triangle sharing two sides $\vec{a}, \vec{b}$: $\frac{1}{2}|\vec{a} \times \vec{b}|$
Volume of a parallelepiped: $\vec{a} \cdot (\vec{b} \times \vec{c})$

1.5 Lines

The **vector form** of a straight line: $\vec{r} = \vec{a} + \lambda\vec{d}$, where $\vec{a}$ is the position vector of a point and $\vec{d}$ is the direction vector.

Parametric form:

$$x = a_1 + \lambda d_1, \qquad y = a_2 + \lambda d_2, \qquad z = a_3 + \lambda d_3$$

Cartesian form:

$$\frac{x - a_1}{d_1} = \frac{y - a_2}{d_2} = \frac{z - a_3}{d_3}$$

When comparing two lines, compare their direction vectors — e.g. if perpendicular, the dot product of the direction vectors equals zero.

1.6 Planes

A plane is fully determined by a point on the plane and a **normal vector** (perpendicular to the plane).

- P_0: a fixed known point on the plane, with position vector $\vec{r_0}$.
- $\vec{n}$: the normal vector — tells you the tilt of the plane.
- P: a variable point on the plane, with position vector $\vec{r}$.
- Since $\vec{n}$ is perpendicular to every vector in the plane: $(\vec{r}-\vec{r_0})\cdot\vec{n} = 0$ for every point in the plane.

Expanding $(\vec{r}-\vec{r_0})\cdot\vec{n} = 0$ component-wise gives

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

which simplifies to the **Cartesian form**

$$ax+by+cz=d$$

where a, b, c come from the normal vector, and d is the dot product of the normal vector with any point on the plane.

A plane can also be described **without** a normal vector — instead using a point and two non-parallel direction vectors lying in the plane:

Parametric form of a plane: $\vec{r} = \vec{r_0} + \lambda\vec{u} + \mu\vec{v}$
where $\vec{u}, \vec{v}$ lie in the plane, $\vec{n} = \vec{u}\times\vec{v}$, and λ, μ are scalar parameters.

1.7 Common Problem Types

Intersection between two lines
Equate $\vec{a}+\lambda\vec{d_1} = \vec{b}+\mu\vec{d_2}$ and solve for λ, μ.

- Unique solution $\Rightarrow$ lines intersect.
- No solution, direction vectors are scalar multiples $\Rightarrow$ parallel.
- No solution, direction vectors are not scalar multiples $\Rightarrow$ skew.

Intersection between a line and a plane
Substitute $\vec{r} = \vec{a}+\lambda\vec{d}$ into $\vec{r}\cdot\vec{n} = \vec{r_0}\cdot\vec{n}$ (or use the Cartesian form $ax+by+cz=d$ and solve for λ).

- One solution $\Rightarrow$ they intersect.
- No solution $\Rightarrow$ parallel.

Intersection between two planes
Using Cartesian form $ax+by+cz=d$:

- Whole equation is a scalar multiple $\Rightarrow$ parallel with all points in common (same plane).

- Coefficients are scalar multiples but constant term isn't $\Rightarrow$ parallel only.
- Otherwise: set one variable to λ, rearrange and solve simultaneously for the others to get the line of intersection $\vec{r} = \vec{a} + \lambda\vec{d}$.

Angle between two lines: $\cos\theta = \dfrac{\vec{d_1}\cdot\vec{d_2}}{|\vec{d_1}||\vec{d_2}|}$

Angle between two planes: $\cos\theta = \dfrac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$

Angle between a line and a plane: $\cos\theta = \dfrac{\vec{d}\cdot\vec{n}}{|\vec{d}||\vec{n}|}$, then take $90^\circ - \theta$ for the actual angle.

Shortest distance: point to plane

Take the normal vector $\vec{n}$ from $ax + by + cz = d$. Using any point on the plane and the given point, form $\vec{d}$. The distance is the scalar projection of $\vec{d}$ onto $\vec{n}$:

$$\frac{\vec{d}\cdot\vec{n}}{|\vec{n}|} = \vec{d}\cdot\hat{n}$$

Alternatively, form a perpendicular line $r = \vec{a} + \lambda\vec{n}$, find its intersection with the plane, then compute the distance between that intersection and the given point.

Shortest distance: skew lines

Find $\vec{n} = \vec{d_1} \times \vec{d_2}$. Take a point from each line to form $\vec{d}$. Distance is the scalar projection of $\vec{d}$ onto $\vec{n}$: $\dfrac{\vec{d}\cdot\vec{n}}{|\vec{n}|}$.

Shortest distance: parallel lines / planes

Lines: For $\vec{a} + \lambda\vec{d_1}$ and $\vec{b} + \mu\vec{d_2}$, form $(\vec{b} - \vec{a}) + \mu\vec{d_2}$. Since this is perpendicular to $\vec{d_2}$, solve $\left[(\vec{b} - \vec{a}) + \mu\vec{d_2}\right]\cdot\vec{d_2} = 0$ for μ, then compute the magnitude. (If given a point instead of a line, use that point directly.)

Planes: Take one point from each plane to form $\vec{d}$; project $\vec{d}$ onto either normal vector $\vec{n}$: $\dfrac{\vec{d}\cdot\vec{n}}{|\vec{n}|}$.

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