The range should be found after the final expression and the final domain are known. Do not use the original range of one component without checking how the composition restricts it.

1.6 Injective, surjective, and inverse workflow

Use this order:

1. Read the domain and codomain.
2. Test injectivity by monotonicity, algebra, or a collision.
3. Test surjectivity by taking an arbitrary codomain element and solving for a preimage.
4. Decide bijectivity.
5. Only then find the inverse.
6. If needed, verify by composition.

For a largest interval problem, do not give a tiny interval just because it works. Find the largest natural restriction on which the function is one-to-one.

2. Proof structure

Proof questions reward clean logic more than long calculation. A good proof begins by naming assumptions and the target.

2.1 Direct proof

For "If A, then B":

1. Assume A.
2. State that the goal is B.
3. Use definitions and algebra to reach B.
4. Finish with a clear therefore sentence.

Skeleton:

Assume A. To show: B. By the definition of ..., ... Therefore B.

2.2 Contrapositive

The contrapositive of $A \Rightarrow B$ is not $B \Rightarrow A$. It is:

$$\text{not } B \Rightarrow \text{not } A.$$

Use this when the conclusion has a divisibility, parity, or impossibility condition that is easier to work with backwards.

2.3 Contradiction

Contradiction starts by assuming the opposite of what is to be shown. It should lead to an impossible statement.

Use it sparingly. If a contrapositive proof is cleaner, use that instead.

2.4 Induction

Induction must have a base case and an induction step.

Skeleton:

1. Let $P(n)$ be the statement ...
2. Base case: prove $P(1)$, or the first value in the problem.
3. Induction step: assume $P(k)$ for an arbitrary k in the range.
4. Prove $P(k+1)$.
5. Conclude by induction.

Be careful about the object being quantified. If the problem says "any collection of objects", do not silently prove it only for the first few special objects.

2.5 Proof phrases worth memorising

Situation	Useful sentence
Arbitrary element	Let x be arbitrary in ...
Existence	Define $x = \ldots$. We verify that ...
Injectivity	Assume $f(x_1) = f(x_2)$. To show: $x_1 = x_2$.
Surjectivity	Let y be arbitrary in the codomain. We seek x in the domain such that $f(x) = y$.
Induction	Assume $P(k)$ is true. To show: $P(k+1)$.
Conclusion	Therefore the required statement holds.