

MATH1061/1961 Linear Algebra Notes

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1 What linear algebra studies

Linear algebra studies finite-dimensional vector spaces and linear transformations between them. In applications, complicated systems are often approximated by linear systems so that questions about geometry, change, equilibrium, and long-run behaviour become questions about vectors and matrices.

2 Vectors, vector spaces and linear combinations

2.1 Vectors in \mathbb{R}^n

A vector in \mathbb{R}^n is an ordered list

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad u_i \in \mathbb{R}. \quad (2.1)$$

Geometrically, vectors can be viewed as floating arrows determined by magnitude and direction, or as position vectors from the origin to a point.

For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$,

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix}, \quad c\mathbf{u} = \begin{bmatrix} cu_1 \\ \vdots \\ cu_n \end{bmatrix}. \quad (2.2)$$