1. ***ignore any dividends;*** or
   - can lead to underestimation
   - distortionary effect on cross-section of stock returns
     - e.g. ignoring dividends → growth stocks favoured more than income/value stocks
2. use adjusted (accounts for div) price series

Simple returns

Given by relative price change:

$$r_t = \frac{p_t - p_{t-1}}{p_{t-1}} \quad (1)$$

📌 EXAMPLE: If $p_{t-1} = 100$ and $p_t = 105$, simple return is

$$r_t = \frac{105 - 100}{100} = 0.05$$

Log returns

or "continuously compounded returns"

$$r_t = \ln \frac{p_t}{p_{t-1}} \quad (3)$$

📌 EXAMPLE: If $p_{t-1} = 100$ and $p_t = 105$, log return is

$$r_t = \ln \frac{105}{100} \approx 0.049 = 4.9\%$$

Small difference between simple and log returns ***when returns are small***

💡 ***Log returns is preferred*** in empirical applications because:

1. naturally interpreted as **"continuously compounding"** returns
   - compounding frequency does not matter → returns are more easily comparable across assets
2. convenient properties of logarithm → they are **time-additive**
   - e.g. weekly log return can be given by sum of daily log returns

Disadvantages of log returns

Typically interested in return on a portfolio (i.e. combo of fin assets)

Simple return on portfolio = weighted avg of simple returns on indiv assets

does not hold for log returns

Log returns → estimate ***value of the portfolio at each time period***, then determine returns for aggregate portfolio value

Using weighted average to calculate portfolio returns

Portfolio simple return

- Variance conditional on the predictor

$$var[r_t|x_{t-1}] = var[a + bx_{t-1} + \epsilon_t|x_{t-1}] = var[\epsilon_t|x_{t-1}] = \sigma^2$$

- Unconditional variance

$$var[r_t] = var[a + bx_{t-1} + \epsilon_t] = b^2 var[x_{t-1}] + \sigma^2$$

- ***Unconditional variance is bigger than conditional***

<u>Conditional heteroskedasticity</u>

We have assumed that ϵ_t is independent of x_{t-1}. If this assumption does not hold, $var[\epsilon_t|x_{t-1}]$ is also given by a function of x_{t-1}

- Example: $\epsilon_t = x_{t-1}u_t$, where u_t is independent of x_{t-1} and $var(u_t) = \sigma_u^2$
- x_{t-1} and ϵ_t are not independent
- $var[\epsilon_t|x_{t-1}] = var[x_{t-1}u_t|x_{t-1}] = x_{t-1}^2 var[u_t|x_{t-1}] = x_{t-1}^2\sigma_u^2$

3.3 Basic portfolio theory

<u>Mean variance analysis of portfolio</u>

A portfolio is a linear combination of individual assets. Portfolio weights add up to 1 (100%) and can be zero, positive, or possibly negative by short-selling.

- ***Short selling:*** *You can borrow shares and selling them to make money first. You must buy back the same number of shares you initially borrowed and return them to the lender later.*
- In this situation if asset price increases in future, return (r_B) on B is "gain". But, return (r_A) on A is regarded as "loss" and it negatively affects the portfolio return.

Mean and variance of a portfolio with 2 stocks: $w_1, w_2 \neq 0$, but $w_3, w_4, \ldots, = 0$

- Notation: $E(r_j) = \mu_j$ and $var(r_j) = \sigma_j^2$
- Note also that $cov(X, Y) = SD(X)SD(Y)corr(X, Y)$

Expected return of a portfolio

$$E(R_p) = E(w_1r_1 + w_2r_2) = w_1\mu_1 + w_2\mu_2 \qquad (17)$$

Variance of a portfolio

$$\begin{aligned} var(R_p) &= var(w_1r_1 + w_2r_2) \\ &= var(w_1r_1) + var(w_2r_2) + cov(w_1r_1, w_2r_1) \\ &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho_{12} \end{aligned}$$

Investors' problem is choosing w_1 and w_2 with the highest expected return and lowest variance

Endogeneity means that $cov(x_t, u_t) = E[x_t u_t] \neq 0$

- problematic & results differ

9.4 Endogeneity

Consider the model: $y_t = \beta_1 + \beta_2 x_{2t} + u_t$

(i) may be violated in presence of stochastic regressors

at times $cov(x_{2t}, u_t) \neq 0$ and this violates the exogeneity assumption, mainly due to the presence of

a. omitted variables
b. measurement errors

<u>Sources of endogeneity</u>

Omitted variables

Suppose the true relationship:

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + u_t,\ cov(x_{jt}, u_t) = 0$$

- but the model is misspecified, $y_t = \beta_1 + \beta_2 x_{2t} + \tilde{u}_t$
- where, x_{3t} is a relevant variable but omitted

Due to misspecification, $\tilde{u}_t = u_t + \beta_3 x_{3t}$

Suppose that $cov(x_{2t}, x_{3t}) \neq 0$, then

$$\begin{aligned} cov(\tilde{u}_t, x_{2t}) &= cov(u_t + \beta_3 x_{3t}, x_{2t}) \\ &= cov(u_t, x_{2t}) + cov(\beta_3 x_{3t}, x_{2t}) \\ &= cov(u_t, x_{2t}) + \beta_3 cov(x_{3t}, x_{2t}) \\ &= 0 + \beta_3 cov(x_{2t}, x_{3t}) \end{aligned}$$

Measurement error

Suppose the true relationship:

$$y_t = \beta_1 + \beta_2 x_{2t} + u_t,\ cov(x_{2t}, u_t) = 0$$

but there exists no perfect measure of x_{2t}

- e.g. in empirical CAPM there is no perfect measure of market portfolio returns, so we may use a specific index fund with observable characteristics

use a proxy since there is no perfect measure

- $\tilde{x}_{2t}$ is not a perfect measure, and thus

$$\tilde{x}_{2t} = x_{2t} + \eta_t$$

where, η_t is the measurement error. That is, we consider the following model:

$$y_t = \beta_1 + \beta_2 \tilde{x}_{2t} + \tilde{u}_t$$

- Due to the presence of measurement error,

$$\tilde{u}_t = u_t - \beta_2 \eta_t$$

$$y_t = \beta_1 + \beta_2(x_{2t} + \eta_t) + u_t$$
$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_2 \eta_t + u_t$$

- Even if the measurement error is uncorrelated with u_t, we have

$$cov(\tilde{u}_t, \tilde{x}_{2t}) = cov(u_t - \beta_2 \eta_t, x_{2t} + \eta_t) = -\beta_2 var(\eta_t) \neq 0$$

Endogeneity bias

Suppose that

$$y_t = \beta_1 + \beta_2 x_{2t} + u_t, \quad cov(x_{2t}, u_t) = E[x_{2t} u_t] = \varrho \neq 0$$

OLS estimator $\hat{\beta}_2 = \frac{\hat{cov}(x_{2t}, y_t)}{\hat{var}(x_{2t})}$

- Under mild cond. these sample versions of covariance and variance converge to true counterparts as sample size gets larger
  - *consistency of the sample covariance*

By the law of large numbers (LLN), the following is true

$$\hat{cov}(x_{2t}, y_t) \to_p cov(x_{2t}, y_t)$$
$$\hat{var}(x_{2t}) \to_p var(x_{2t})$$

where, $\to_p$ denotes *convergence in probability*

Therefore,

$$\hat{\beta}_2 \to_p \frac{cov(x_{2t}, y_t)}{var(x_t)}$$

However, note that

$$cov(x_{2t}, y_t) = cov(x_{2t}, \beta_1 + \beta_2 x_{2t} + u_t)$$
$$= cov(x_{2t}, \beta_2 x_{2t}) + cov(x_{2t}, u_t)$$
$$= \beta_2 var(x_{2t}) + cov(x_{2t}, u_t)$$

$cov(x_{2t}, u_t) \neq 0$ (due to endogeneity)

Therefore,

$$\frac{cov(x_{2t}, y_t)}{var(x_t)} = \beta_2 + \frac{cov(x_{2t}, u_t)}{var(x_{2t})} \neq \beta_2$$

where, term in **red** is the ***endogeneity bias***

> **What happened?**
>
> As the sample size gets larger, $\hat{\beta}_2$ gets closer to something that is not equal to β_2
>
> In this case, we say that $\hat{\beta}_2$ **is inconsistent**
>
> Consequently, $\hat{\beta}_1$ is inconsistent as well
>
> - since OLS estimator for β_1 is a function of β_2

📌 EXAMPLE: a simple form of the efficient market hypothesis

$$E[P_{t+1}|\Omega_T] = P_t$$

the reasonable prediction of future asset price using all available info. up to time t, must be currnet price

*Stochastic process (time series) P_t satisfying above is a **martingale** (w. resp. to info. set Ω_t)*

EXAMPLE of martingale: **Random walk**

$$X_{t+1} = X_t + \epsilon_{t+1},$$

assume $E[\epsilon_{t+1}|\Omega_t] = 0$

and $X_t \in \Omega_t$

Then, $E[X_{t+1}|\Omega_t] = E[X_t|\Omega_t] + E[\epsilon_{t+1}|\Omega_t] = X_t$

---

11. Time series prediction and inference

11.1 Autoregressive predictive models

Often, time series prediction is implemented by autoregressive (AR) models

- AR predictive model assumes that ***future value can be predicted from past*** (i.e. lagged variables)

Statistical properties of stationary AR models

Consider the model

$$x_t = \phi_0 + \phi_1 x_{t-1} + \dots + \phi_p x_{t-p} + u_t$$

where,

$$E(u_t) = 0$$

$$var(u_t) = \sigma^2$$

$$cov(u_t, x_{t-k}) = 0, k \geq 1$$

Mean of a stationary AR process

$$E(x_t) = \phi_0 + \phi_1 E(x_{t-1}) + \phi_2 E(x_{t-2}) + \dots + \phi_p E(x_{t-p}) + \underbrace{E(u_t)}_{=0}$$

strict/weak stationarity implies:

$$E(x_t) = E(x_{t-1}) = \dots = E(x_{t-p}) = \mu_X$$

thus, $\mu_X = \phi_0 + \phi_1 \mu_X + \phi_2 \mu_X + \dots + \phi_p \mu_X$

$$\mu_X = \frac{\phi_0}{1 - \phi_1 - \phi_2 - \dots - \phi_p}$$

Variance of a stationary AR process

In the case of $p = 1$,

$$var(x_t) = var(\phi_0 + \phi_1 x_{t-1} + u_t)$$
$$= \phi_1^2 var(x_{t-1}) + \sigma^2$$

weak stationarity implies:

$$var(x_t) = var(x_{t-1}) = \sigma_X^2$$

thus, $\sigma_X^2 + \phi_1^2 \sigma_X^2 + \sigma^2$

$$\sigma_X^2 = \frac{\sigma^2}{1 - \phi_1^2}$$

📌 **Variance of x_t is inflated by the persistence of (ϕ_1) of the AR model**

when $\phi = 0.9$ and

$x_t = 0.9 x_{t-1} + u_t$ means that $\sigma_X^2 > \sigma^2$

when $\phi = 0.1$ and

$x_t = 0.1 x_{t-1} + u_t$ means that $\sigma_X^2 \gtrapprox \sigma^2$

Autocovariances of a stationary $AR(p)$ process

The autocovariances and autocorrelation functions can be computed by solving the Yule-Walker equations:

The ***Yule-Walker equations*** are given by

$$\gamma_0 = \phi_1 \gamma_1 + \phi_2 \gamma_2 + \sigma^2$$
$$\gamma_1 = \phi_1 \gamma_0 + \phi_2 \gamma_1$$
$$\gamma_2 = \phi_1 \gamma_1 + \phi_2 \gamma_0$$

.

.

.

$$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_k$$

Autocovariances $\gamma_1, \ldots, \gamma_k$ can be computed