PDEs and Waves

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The Heat Equation (Ring)

By way of motivation, suppose we want to solve the following PDE-IBVP modelling heat distribution on a metal ring of circumference 2π:

$$u_t = \alpha\, u_{xx}, \qquad -\pi \le x \le \pi,\ t > 0, \quad \alpha > 0,$$

subject to the periodic boundary conditions

$$u(-\pi, t) = u(\pi, t), \qquad u_x(-\pi, t) = u_x(\pi, t),$$

and the initial condition

$$u(x, 0) = f(x).$$

For simplicity we set $\alpha = 1$. Assume a separated form $u(x,t) = X(x)T(t)$. Then

$$X(x)T'(t) = X''(x)T(t) \quad \Longrightarrow \quad \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} =: -\lambda,$$

where $\lambda \in \mathbb{R}$ is a constant. Hence

$$T'(t) = -\lambda T(t), \qquad X''(x) + \lambda X(x) = 0.$$

The time part solves $T(t) = C\, e^{-\lambda t}$. The space part is an eigenvalue problem with periodic boundary conditions

$$X''(x) + \lambda X(x) = 0, \qquad X(-\pi) = X(\pi), \quad X'(-\pi) = X'(\pi).$$

We consider three cases for λ.

(i) $\lambda < 0$. Writing $\lambda = -\mu^2$ with $\mu > 0$, the general solution $X(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}$ cannot satisfy both periodic conditions unless $C_1 = C_2 = 0$. Thus there is no nontrivial periodic eigenfunction.

(ii) $\lambda = 0$. Then $X(x) = A_1 x + A_0$. Periodicity forces $A_1 = 0$, so $X \equiv A_0$ is an eigenfunction.

(iii) $\lambda > 0$. Write $\lambda = k^2$ with $k > 0$. Then

$$X(x) = A\cos(kx) + B\sin(kx).$$

Thus, under these boundary conditions the heat equation admits two families of solutions:

$$u_k(x,t) = e^{-k^2 t}\cos(kx), \qquad v_k(x,t) = e^{-k^2 t}\sin(kx), \qquad k = 0, 1, 2, \ldots$$

By the superposition principle and Ignoring convergence issues for the moment, any function of the form

$$u(x,t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \Big(a_k u_k(x,t) + b_k v_k(x,t) \Big)$$

Which is

$$u(x,t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \Big(a_k e^{-k^2 t}\cos(kx) + b_k e^{-k^2 t}\sin(kx) \Big),$$

At $t = 0$ we require

$$u(x,0) = f(x) \quad \Longrightarrow \quad f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \big(a_k \cos(kx) + b_k \sin(kx) \big).$$

The question is: what kind of functions $f(x)$ can be expressed as a (potentially infinite) linear combination of sines and cosines? The answer is: essentially all functions (at least in a suitable sense).

Subsection 5.2

Fourier Series

Definition 1 Let $f(x)$ be a 2π-periodic function which is piecewise continuous on $[-\pi, \pi]$. Then f can be expanded (at least formally) into a Fourier series of the form

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos(nx) + b_n \sin(nx) \Big),$$

where the Fourier coefficients are given by

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx, \qquad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)\,dx, \quad n \geq 1.$$

Lemma 1 If $k, \ell > 0$, then

$$\int_{-\pi}^{\pi} \cos(kx)\cos(\ell x)\,dx = \begin{cases} 0, & k \neq \ell, \\ \pi, & k = \ell, \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(kx)\sin(\ell x)\,dx = \begin{cases} 0, & k \neq \ell, \\ \pi, & k = \ell, \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(kx)\cos(\ell x)\,dx = 0, \quad \forall k, \ell.$$

Example Find the Fourier series of the function

$$f(x) = x^2 - \pi^2, \quad x \in [-\pi, \pi].$$

Proof By linearity, it suffices to compute the Fourier series of x^2 on $[-\pi, \pi]$ and then subtract π^2. For x^2, the Fourier coefficients are

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2\,dx = \frac{2}{\pi} \int_{0}^{\pi} x^2\,dx = \frac{2\pi^2}{3},$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(kx)\,dx = \frac{2}{\pi} \int_{0}^{\pi} x^2 \cos(kx)\,dx = \frac{4(-1)^k}{k^2} \quad (k \geq 1),$$

and, since x^2 is even,

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin(kx)\,dx = 0 \quad (k \geq 1).$$

Hence

$$x^2 \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) = \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos(kx).$$

Therefore,

$$f(x) = x^2 - \pi^2 \sim \left(\frac{\pi^2}{3} - \pi^2 \right) + 4\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos(kx) = -\frac{2\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos(kx).$$

□

Example Compute the Fourier series of the function $f(x) = x$ on the interval $[-\pi, \pi]$.

Proof We compute the Fourier coefficients. First,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = 0.$$

Next, since $x\cos(kx)$ is an odd function on $[-\pi, \pi]$, we have

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(kx)\,dx = 0.$$

Finally,

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(kx)\,dx.$$

Integrating by parts, we obtain

$$b_k = \frac{1}{\pi}\left[\frac{x\cos(kx)}{k} + \frac{\sin(kx)}{k^2}\right]_{-\pi}^{\pi} = \frac{2}{k}(-1)^{k+1}.$$

Therefore, the Fourier series of $f(x) = x$ is

$$x \sim 2\left(\sin x - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \frac{\sin(4x)}{4} + \cdots\right).$$

□

Remark Regarding convergence: standard tests such as the ratio and root tests are inconclusive. Nevertheless, the series does converge on $[-\pi, \pi]$, although at the endpoints we observe

$$\text{Series at } x = \pm\pi \;\Rightarrow\; 0, \qquad f(\pm\pi) = \pm\pi.$$

Definition 2 Let

$$s_N(x) := \frac{a_0}{2} + \sum_{k=1}^{N}\big[a_k\cos(kx) + b_k\sin(kx)\big]$$

be the N-th partial sum of a Fourier series. We say the Fourier series *converges at a point x* if the limit

$$\lim_{N\to\infty} s_N(x) = \lim_{N\to\infty}\left(\frac{a_0}{2} + \sum_{k=1}^{N}[a_k\cos(kx) + b_k\sin(kx)]\right)$$

exists and equals some number L. If, in addition, $L = f(x)$, we say the Fourier series *converges to f at x*.

Remark

$$s_N(x) = \frac{a_0}{2} + \sum_{k=1}^{N}\big[a_k\cos(kx) + b_k\sin(kx)\big],$$

is called a *trigonometric polynomial* because, by judicious (and repeated) applications of trigonometric identities, it can be expressed as a polynomial in the functions $\cos x$ and $\sin x$.

Conversely, every such polynomial can be expanded as a sum of sines and cosines of increasing frequency.

Remark The functions $\cos kx$ and $\sin kx$ are all periodic functions of period 2π, so if a Fourier series converges, the limiting function $\tilde{f}(x)$ must also be periodic with the same period.

Thus, it is unreasonable to expect that the Fourier series of a function like $f(x) = x$, which is not periodic on $[-\pi, \pi]$, converges everywhere. Instead, one should expect it to converge to its *periodic extension*.

Definition 3 If $f(x)$ is any function defined on $-\pi \le x \le \pi$, then there is a unique 2π-periodic function, called the 2π-*periodic extension* of f and denoted by $\tilde{f}(x)$, such that

$$\tilde{f}(x) = f(x), \quad -\pi < x < \pi,$$

and

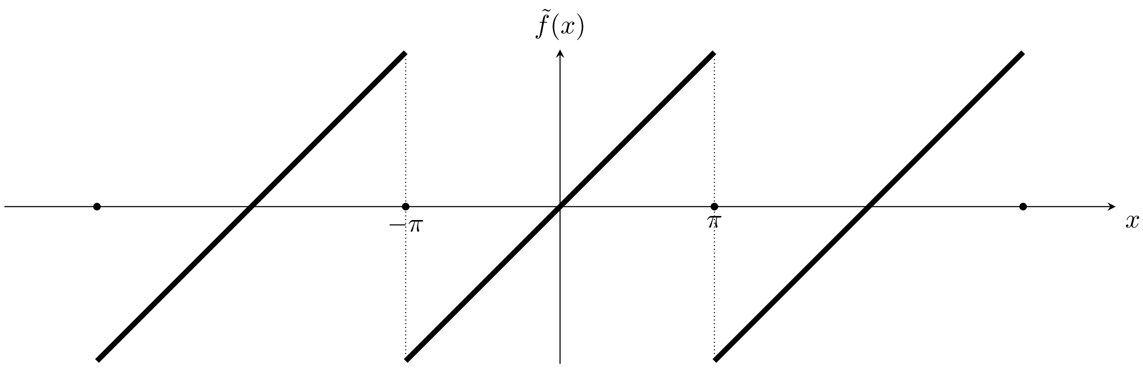
$$\tilde{f}(\pm\pi) = \frac{f(\pi) + f(-\pi)}{2}.$$

That is, $\tilde{f}(x)$ equals $f(x)$ on the interior of the interval and takes the average value of f at the endpoints.

Example The 2π-periodic function of $f(x) = x$ is the *sawtooth function*, which takes the value 0 at odd integer multiples of π (in fact, at all integer multiples of π, but for different reasons). Explicitly,

$$\tilde{f}(x) = \begin{cases} x - 2m\pi, & (2m-1)\pi < x < (2m+1)\pi, \\ 0, & x = (2m-1)\pi, \end{cases}$$

for $m \in \mathbb{Z}$.
Thus, $\tilde{f}(x)$ is the 2π-periodic extension of $f(x) = x$.



Definition 4 A function $f(x)$ is called *piecewise* C^1 on an interval $[a, b]$ if it is defined, continuous, and continuously differentiable except at a **finite number** of points

$$a \le x_1 \le x_2 \le \cdots \le x_n \le b.$$

At each exceptional point, the function and its derivative can have at worst a *jump discontinuity*. That is, the left- and right-hand limits exist:

$$f(x_k^-) = \lim_{x \to x_k^-} f(x), \quad f(x_k^+) = \lim_{x \to x_k^+} f(x),$$

$$f'(x_k^-) = \lim_{x \to x_k^-} f'(x), \quad f'(x_k^+) = \lim_{x \to x_k^+} f'(x).$$

For a piecewise C^1 function, an exceptional point is either a *jump discontinuity*, where the left- and right-hand derivatives exist, or a *corner*, meaning a point where f is continuous so that $f(x_k^-) = f(x_k^+)$, but has different left- and right-hand derivatives.

Theorem 5.1 If $\tilde{f}(x)$ is a 2π-periodic, piecewise C^1 function, then at any $x \in \mathbb{R}$ its Fourier series converges to $\tilde{f}(x)$ if $\tilde{f}$ is continuous at x, and converges to

$$\frac{1}{2}\left(\tilde{f}(x^+) + \tilde{f}(x^-)\right)$$

(the average of the left and right limits) if x is a jump discontinuity.
Thus, the Fourier series of a function defined and piecewise C^1 on $-\pi \leq x \leq \pi$ converges to the 2π-periodic extension wherever said extension is continuous, and to the average of the jumps at the jump discontinuities.

Example For example $f(x) = x$, we can replace the $\sim$ with an equality sign provided we are in the interior of the interval, i.e. $x \in (-\pi, \pi)$. Plugging in $x = \frac{\pi}{2}$ and observing that

$$\sin\big((2k+1)\pi/2\big) = (-1)^k,$$

equation becomes

$$\frac{\pi}{2} = 2\left(\sin\frac{\pi}{2} - \frac{\sin\pi}{2} + \frac{\sin\frac{3\pi}{2}}{3} - \frac{\sin 2\pi}{4} + \cdots\right),$$

$$\frac{\pi}{2} = 2\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right).$$

Rearranging, we get a (very slowly converging) series for π. Namely, that 4 times the alternating sum of the reciprocals of the odd numbers is π. That is,

$$\pi = 4\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1} = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right).$$

This series and its sum has, in some form or another, been known to humans for roughly 500 years, well before Fourier series.

Remark To compute a Fourier series for a function initially defined on $[0, \pi]$, one standard approach is to extend it to a 2π-periodic function on $\mathbb{R}$. Two efficient extensions to $[-\pi, \pi]$ are obtained by using even and odd reflections, which often simplify the Fourier coefficients.

Definition 5 A function $f(x)$ is called *odd* if $f(-x) = -f(x)$, and it is called *even* if $f(-x) = f(x)$.

Lemma 2

1. The sum of two even functions is even, and the sum of two odd functions is odd.
2. The product of two even functions or two odd functions is even, while the product of an even function and an odd function is odd.

Lemma 3 If $f(x)$ is odd and integrable on the interval $[-a, a]$, then $\int_{-a}^{a} f(x), dx = 0$. If $f(x)$ is even and integrable, then $\int_{-a}^{a} f(x), dx = 2\int_{0}^{a} f(x), dx$.

Remark These facts will immediately simplify the computation of Fourier coefficients for 2π-periodic extensions.

Proposition 1 If $f(x)$ is even, then the coefficients of the sine functions in its Fourier expansion are all 0, so $f(x)$ can be represented by a Fourier cosine series:

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx,$$

where

$$a_k = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(kx)\, dx, \quad k = 0, 1, 2, \ldots$$

Proposition 2 If $f(x)$ is odd, then the Fourier cosine coefficients are all 0, so $f(x)$ can be represented by a Fourier sine series:

$$f(x) \sim \sum_{k=1}^{\infty} b_k \sin kx,$$

where

$$b_k = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(kx)\, dx, \quad k = 1, 2, \ldots$$

Remark Conversely, a convergent Fourier sine series always represents an odd function, while a convergent Fourier cosine series always represents an even function.

Example The absolute value function $|x| = f(x)$ is an even function, so $b_k = 0$ and it has a Fourier cosine series. Its coefficients are given by

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x\, dx = \pi$$

and

$$a_k = \frac{2}{\pi} \int_0^{\pi} x \cos kx\, dx = \begin{cases} 0 & k \neq 0 \text{ even}, \\ -\dfrac{4}{k^2 \pi} & k \text{ odd}. \end{cases}$$

We can also use the convergence theorem to get another number theoretic result for this Fourier series. We have

$$|x| \sim \frac{\pi}{2} - \frac{4}{\pi} \left(\cos x + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + \cdots \right).$$

And we can write if $x \in [-\pi, \pi]$. More specifically, if we substitute $x = 0$, we can re-arrange to get

$$\frac{\pi^2}{8} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \left(1 + \frac{1}{9} + \frac{1}{25} + \cdots \right).$$

Thus, the sum of the reciprocals of the odd squares is $\pi^2/8$.

Remark It is worth noting that any function on $[0, \pi]$, whether it be even or odd, will have an even or odd extension to $[-\pi, \pi]$. For example if $f(x) = \sin x$ on $[0, \pi]$, then its Fourier cosine series coefficients are given as

$$a_k = \frac{2}{\pi} \int_0^{\pi} \sin x \cos kx\, dx = \begin{cases} \dfrac{4}{(1-k^2)\pi}, & k \text{ even}, \\ 0, & k \text{ odd}. \end{cases}$$

So we have

$$|\sin x| \sim \frac{2}{\pi} - \frac{4}{\pi} \sum_{j=1}^{\infty} \frac{\cos 2jx}{4j^2 - 1}.$$

On the other hand $\sin x$ is already odd, and so its odd periodic extension is just $\sin x$, which has Fourier series $\sin x$ for the same reason 1 had Fourier series 1.

Subsection 5.3
Complex Fourier series

Finally, it is often convenient to exploit Euler's formula and use complex exponentials instead of sines and cosines. Because we have

$$e^{ikx} = \cos kx + i \sin kx, \quad e^{-ikx} = \cos kx - i \sin kx$$

or

$$\cos kx = \frac{e^{ikx} + e^{-ikx}}{2}, \quad \sin kx = \frac{e^{ikx} - e^{-ikx}}{2i}.$$

So if we have a series representation of $f(x)$ in terms of sines and cosines, we can get one in terms of complex exponentials and vice versa.

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} \frac{(a_k - ib_k)}{2} e^{ikx} + \frac{(a_k + ib_k)}{2} e^{-ikx}$$
$$\sim \sum_{k=-\infty}^{\infty} c_k e^{ikx}.$$

We note that as $c_n = \frac{1}{2}(a_n + ib_n)$, we have, as we've defined a_n and b_n via integration against f, that

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} \, dx.$$

Example Write the *complex* Fourier series on $[-\pi, \pi]$ for:

(a) $f(x) = x^2 + 1$,

(b) $w(x) = \sum_{n=0}^{\infty} \frac{\sin(2^n x)}{2^n}$.

Proof **(a):** For $f(x) = x^2 + 1$,

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 + x^2) \, dx = 1 + \frac{\pi^2}{3},$$

and for $n \neq 0$,

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 + x^2) e^{-inx} \, dx = \frac{2(-1)^n}{n^2}.$$

So

$$f(x) \sim 1 + \frac{\pi^2}{3} + \sum_{n \neq 0} \frac{2(-1)^n}{n^2} e^{inx}.$$

(b): For $w(x) = \sum_{n=0}^{\infty} \frac{\sin(2^n x)}{2^n}$, we note this is already a Fourier sine series with only frequencies $k = 2^n$. Recall

$$c_k = \tfrac{1}{2}(a_k - ib_k), \qquad c_{-k} = \tfrac{1}{2}(a_k + ib_k).$$

Here $a_k = 0$ for all k, and $b_k = 0$ unless $k = 2^n$. Thus

$$c_{2^n} = -\frac{i}{2^n}, \qquad c_{-2^n} = \frac{i}{2^n}.$$

Hence

$$w(x) \sim \sum_{n=0}^{\infty} \left(c_{2^n} e^{i2^n x} + c_{-2^n} e^{-i2^n x} \right) = \sum_{n=0}^{\infty} \frac{e^{i2^n x}}{i \, 2^{n+1}}.$$

□

SUBSECTION 5.4

Change of scale

So far all of the Fourier series we have dealt with have been for functions on the interval $[-\pi, \pi]$, but it is natural to sometimes want to compute Fourier series for functions defined on a different interval, say $[-L, L]$. In this case you can perform the change of variables $y = \frac{\pi x}{L}$ so that $y \in [-\pi, \pi]$ when $x \in [-L, L]$. Then for a function $f(x)$ on $[-L, L]$ the rescaled function $F(y) = f\left(\frac{L}{\pi}y\right)$ will be defined on $[-\pi, \pi]$.

If we can compute the Fourier series of F:

$$F(y) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos ky + b_k \sin ky,$$

then we can revert to the original x variable by substituting in $y = \frac{\pi}{L}x$ and get the Fourier series for F:

$$f(x) = f\left(\tfrac{\pi}{L} \cdot \tfrac{L}{\pi}x\right) = F\left(\tfrac{\pi}{L}x\right) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\tfrac{k\pi x}{L}\right) + b_k \sin\left(\tfrac{k\pi x}{L}\right).$$

The Fourier coefficients are

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} F(y)\cos ky\, dy, \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} F(y)\sin ky\, dy.$$

Again, we can make the substitution that $y = \frac{\pi}{L}x$ so $dy = \frac{\pi}{L}dx$ in the integral and we get the formula for the rescaled Fourier coefficients in the original x variables:

$$a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\tfrac{k\pi x}{L}\right) dx, \quad b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\tfrac{k\pi x}{L}\right) dx.$$

For many practical purposes, this is the formula you want to remember, so we separate it out as a theorem.

Theorem 5.2 Let $f(x)$ be an integrable function on $[-L, L]$. The Fourier series of the $2L$ periodic extension of $f(x)$ is

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right),$$

where

$$a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{k\pi x}{L}\right) dx, \qquad b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{k\pi x}{L}\right) dx.$$

Likewise, the complex Fourier series on $[-L, L]$ is

$$f(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{\frac{ik\pi x}{L}}, \qquad c_k = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{\frac{ik\pi x}{L}}\, dx.$$

Subsection 5.5

Integration and Differentiation

5.5.1 Integration of Fourier Series

Let's start this section with an observation. The zeroth coefficient of the Fourier series of a function $f(x)$ on $[-\pi, \pi]$ is found by

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

This says that the *mean* or average value of the function $f(x)$ on the interval $[-\pi, \pi]$ is its zeroth Fourier coefficient (this fact is unchanged on any symmetric interval $[-L, L]$).

Suppose now we start with an integrable function, $f(x)$ defined on $[-\pi, \pi]$ and we write the Fourier series of its 2π periodic extension

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}\big(a_n\cos(nx) + b_n\sin(nx)\big).$$

We'd like to investigate its integral

$$g(x) = \int_0^x f(y)\,dy.$$

Naively we'd expect to be able to integrate the series term by term, or to interchange the limits of integration and summation (and indeed this is the case for some mild conditions on f as we shall see shortly). We have

$$\begin{aligned} g(x) = \int_0^x f(y)\,dy &\sim \int_0^x\Big(a_0 + \sum_{n=1}^{\infty}(a_n\cos(ny) + b_n\sin(ny))\Big)\,dy \\ &\sim a_0x + \sum_{n=1}^{\infty}\left(a_n\int_0^x\cos(ny)\,dy + b_n\int_0^x\sin(ny)\,dy\right) \\ &\sim a_0x + \sum_{n=1}^{\infty}\left(\frac{a_n}{n}\sin(nx) - \frac{b_n}{n}\cos(nx)\right) + \sum_{n=1}^{\infty}\frac{b_n}{n}. \end{aligned}$$

Let's pause for a moment and analyse the last expression:

$$g(x) \sim a_0x + \sum_{n=1}^{\infty}\left(\frac{a_n}{n}\sin(nx) - \frac{b_n}{n}\cos(nx)\right) + \sum_{n=1}^{\infty}\frac{b_n}{n}.$$

The first thing to notice is that it is not a trigonometric series if a_0 is not zero. If a_0 is non-zero we can do one of two things, we can bring the a_0x over to get a Fourier series for $g(x) - a_0x$:

$$g(x) - a_0x \sim \sum_{n=1}^{\infty}\left(\frac{a_n}{n}\sin(nx) - \frac{b_n}{n}\cos(nx)\right) + \sum_{n=1}^{\infty}\frac{b_n}{n}.$$

Remark Notice that in light of the observation opening this section, that this says that the mean of the function $\int_0^x f(y)\,dy - a_0x$ is given by

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\int_0^x (f(y) - a_0)\,dy\right)dx = \sum_{n=1}^{\infty}\frac{b_n}{n}.$$

Further if we observe that $\int_0^x a_0\,dy = a_0x$ is an odd function and so has mean zero, we have that the mean of

$$g(x) = \int_0^x f(y)\,dy$$

itself is given by $\frac{1}{n}$ times the Fourier sine coefficients of its derivative f:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\int_0^x f(y)\,dy\right)dx = \sum_{n=1}^{\infty}\frac{b_n}{n}.$$

The other thing we could potentially do is to replace the function x with the Fourier series of the 2π periodic extension of x on $[-\pi,\pi]$. We have already seen that for $x \in [-\pi,\pi]$ we have

$$x \sim 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin(nx).$$

So substituting this in to the last expression we have that the previous equation becomes:

$$g(x) \sim \sum_{n=1}^{\infty}\frac{b_n}{n} + \sum_{n=1}^{\infty}\left(a_n + \frac{2a_0(-1)^{n-1}}{n}\right)\sin(nx) + \sum_{n=1}^{\infty}\frac{b_n}{n}\cos(nx).$$

Theorem 5.3 If f is a piecewise continuous function with mean 0 on the interval $[-\pi,\pi]$, then its Fourier series

$$f(x) \sim \sum_{k=1}^{\infty} a_k\cos(kx) + b_k\sin(kx)$$

can be integrated term by term to produce the Fourier series

$$g(x) = \int_0^x f(y)\,dy \sim m + \sum_{k=1}^{\infty}\left(-\frac{b_k}{k}\cos kx + \frac{a_k}{k}\sin kx\right).$$

The 0^{th} term

$$m = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\,dx$$

is the mean of the integrated function.

Remark Even though we've restricted the previous theorem to mean zero functions, this can be avoided by defining a new function $\tilde{f}(x) = f(x) - a_0$, which is then mean zero, and then applying the theorem to it. Rearranging and substituting in the Fourier series for x will produce the Fourier series of the integral of the original $f(x)$.

Example The function $f(x) = x$ does have mean zero, so we're in the (slightly) simpler case and

$$\frac{x^2}{2} = \int_0^x y\,dy \sim -2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}\cos(nx) + 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}.$$

Let's integrate again and see what we get

$$\frac{x^3}{6} \sim \frac{\pi^2}{6}x - 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}\sin(nx).$$

Now again, we get something that isn't a Fourier series, but again, we can replace the x with the Fourier series of its 2π periodic extension to get

$$\begin{aligned}\frac{x^3}{6} &\sim 2\sum_{n=1}^{\infty}\left(\frac{\pi^2(-1)^{n-1}}{6n} - \frac{(-1)^{n-1}}{n^3}\right)\sin(nx)\\ &\sim \sum_{n=1}^{\infty}\frac{\pi^2 n^2 - 6}{3n^3}(-1)^{n-1}\sin(nx),\end{aligned}$$

which you can (and should) check is indeed the Fourier series for the function $\frac{x^3}{6}$.

5.5.2 Differentiation of Fourier Series

Something quite different happens when we try to differentiate Fourier series term-wise though. Let's start with the Fourier series of x.

$$x \sim 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin(nx).$$

Differentiating the left-hand side and the right-hand side term by term, we are led to the following

$$1 \sim 2\sum_{n=1}^{\infty}(-1)^{n-1}\cos(nx). \tag{5.1}$$

There are a couple of problems with this expression. First, the right hand side is not the Fourier series of 1. We already know that the Fourier series of 1 is just the constant function 1. Secondly, looking at $x = 0$, we get the seemingly meaningless expression

$$1 \sim 2(1 - 1 + 1 - 1 + 1 - \cdots).$$

At $\pm\pi$ things are even worse. We have that $\cos(\pm n\pi) = (-1)^n$ so the series reads

$$1 \sim -2(1 + 1 + 1 + 1 + \cdots).$$

Indeed, the right hand side does not converge for any value of x in the interval $[-\pi, \pi]$. So what is going on? Well, first of all, the right hand side of (5.1) is not the Fourier series of x, but rather the sawtooth function, which is not continuous at the points $(2k + 1)\pi$ with $k \in \mathbb{Z}$. So the derivative is 1 everywhere except at odd multiples of π, where it is undefined.

The Fourier series for x will show up in the integral for any function with mean not equal to 0. This means that integrating a function with mean not equal to zero and then differentiating will produce this mess. But surely not all Fourier series "break" like the previous example. So we have the following:

Theorem 5.4 Let $f(x)$ have a piecewise C^2 and continuous 2π-periodic extension, then its Fourier series can be differentiated term by term to produce the Fourier series of its derivative

$$f'(x) \sim \sum_{k=1}^{\infty} \left(kb_k \cos(kx) - ka_k \sin(kx)\right).$$

Subsection 5.6

Convergence of Fourier Series

Definition 1 We say that the sequence of functions $f_n(x) : I \to \mathbb{R}$ **converges pointwise** at $x = a \in I$ if the limit of $f_n(a)$ as $n \to \infty$ exists (and is finite), i.e.

$$\lim_{n\to\infty} f_n(a) = L.$$

In particular, this means that for each $\varepsilon > 0$ there exists an $N = N(\varepsilon, a)$ such that if $n > N$ then

$$|f_n(a) - L| < \varepsilon.$$

If for every $a \in I$ we have that the sequence f_n converges pointwise, say

$$\lim_{n\to\infty} f_n(a) = L_a,$$

then we can define a new function $f_L(x)$ on I where $f_L(a) = L_a$. In this case we say that the sequence $f_n(x)$ converges pointwise on I to $f_L(x)$.

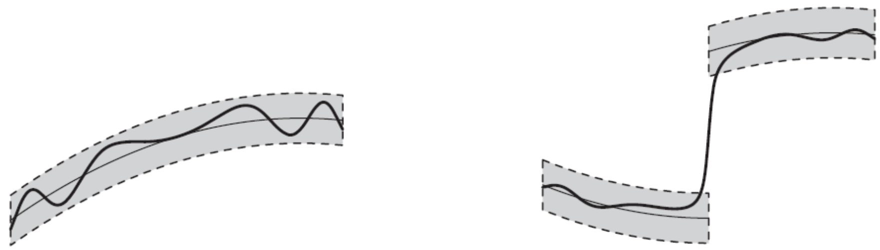


Figure 2. Uniform and non uniform convergence

In the previous, we saw that in order to get the sequence $f_n(a)$ close to its limit value, the number N we chose depended on both the accuracy we required and the point a we were looking at. It is not always necessary that the situation should be like this. In particular, if we are lucky, we can find that for each $a \in I$ the same value of N works. This is called **uniform convergence** of the sequence of functions on I.

Geometrically, this means that the sequence of functions is collapsing into a narrow tube around the limiting function f_L, and this tube works for *all* $x \in I$. (See Figure 2 on the left.)

Definition 2 We say that a sequence of functions $f_n(x) : I \to \mathbb{R}$ **converges uniformly** on the interval I, to the function $f_L(x)$ if for each $\varepsilon > 0$ we have an $N = N(\varepsilon)$ (that depends only on ε), such that for *every* $a \in I$, if $n > N$ we can write

$$|f_n(x) - f_L(x)| < \varepsilon.$$

The key difference here is that for each ε we must find a single N that works for all $a \in I$. We notice that uniform convergence implies pointwise convergence, but not the other way round. As we saw in Example 4.1, pointwise convergence can occur for a sequence of continuous functions converging to a discontinuous function, but we cannot have uniform convergence of continuous functions to a discontinuous function. The reason is that if the jump discontinuity is larger than ε, then the εtube around the limit function near the discontinuity will be disconnected, and so no continuous function can lie in the εtube on both sides of the jump. (See Figure 2 on the right.) In fact, this is one of the main features of uniform convergence: it preserves continuity in the limit.

Theorem 5.5 If $f_n(x) : I \to \mathbb{R}$ is a sequence of continuous functions and f_n converges uniformly on I to $f_L(x)$, then $f_L(x)$ is continuous.

In fact this theorem is very powerful, and often used in contrapositive form. That is, if a sequence of continuous functions converges pointwise to a discontinuous function, then it **cannot** converge uniformly to that function.

However, if we change the interval, avoiding the point of discontinuity of the limit function, then we may have uniform convergence.

Example Consider the sequence $f_n(x) = x^n$ on $[0, 1]$, which converges pointwise to the discontinuous limit

$$f(x) = \begin{cases} 0, & 0 \le x < 1, \\ 1, & x = 1. \end{cases}$$

This convergence is not uniform on $[0, 1]$ because of the discontinuity at $x = 1$. However, if we change the interval to $[0, 0.9]$, then f_n converges uniformly to 0.

Example Consider the sequence of functions $f : [0,5] \to \mathbb{R}$ given by

$$f_n(x) = \frac{2nx}{1+n^2x^2}.$$

Proof We first note that $f_n(0) = 0$ for all n, and for any fixed $x = a \in (0,5]$ we have

$$f_n(a) = \frac{2na}{1+n^2a^2} \le \frac{2na}{n^2a^2} \le \frac{2}{na}.$$

To guarantee that $f_n(a) < \varepsilon$, we need to choose

$$N = \frac{2}{a\varepsilon}.$$

Then for $n > N$ we obtain

$$f_n(a) < \frac{2}{na} < \varepsilon.$$

Thus $\{f_n\}$ converges pointwise to 0 on $[0,5]$.
However, observe that here

$$N = N(\varepsilon, a) = \frac{2}{a\varepsilon},$$

so N depends on both ε and a. This shows we do not have uniform convergence. Indeed, at $x = \frac{1}{n}$ we find

$$f_n\left(\tfrac{1}{n}\right) = 1.$$

Therefore, if $\varepsilon < 1$, the sequence $\{f_n\}$ cannot stay within ε of 0 for all $x \in [0,5]$. Hence the convergence is pointwise but not uniform. □

However, you do have the following theorem

Theorem 5.6 Suppose that $f_n : I \to \mathbb{R}$ is a sequence of integrable functions on $I = [a,b]$ and that $f_n(x) \to f(x)$ uniformly on $[a,b]$. Then the limit function $f(x)$ is integrable on $[a,b]$ and

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left(\int_a^b f_n(x)\,dx\right) = \int_a^b \left(\lim_{n\to\infty} f_n(x)\,dx\right).$$

That is, you can integrate the sequence and the sequence of integrals converges to the integral of the limit. This is actually not totally necessary as we have seen. We can integrate a Fourier sequence term-wise which does not converge uniformly and still produce the Fourier series of the integral as we've seen in theorem 5.2

What about differentiation? Well, there things are a bit more complicated. Here you need pointwise convergence of the sequence, and uniform convergence of the sequence of derivatives.

Theorem 5.7 Suppose that $f_n : I \to \mathbb{R}$ is a sequence of C^1 functions on an interval $I = [a,b]$ and that $f_n(x) \to f(x)$ pointwise for each $x \in I$. If $f_n'(x) \to \phi$ uniformly on $[a,b]$, then the function f is differentiable and $f'(x) = \phi(x)$ on $[a,b]$. Moreover, $f_n \to f$ uniformly on $[a,b]$.

This is a slightly stronger requirement than what is in theorem 5.3, as we shall see. At this point we have covered uniform/pointwise convergence of sequences of functions, but we're interested in the convergence of *series* of functions, in particular Fourier series. Well, the convergence (uniform, or pointwise, or other) of an infinite series of functions is by definition defined in terms of the *sequence* of partial sums of that series

$$s_n(x) = \sum_{k=1}^{n} f_k(x).$$

That is if $s_n(x)$ converges uniformly or pointwise to a limiting function then the series is said to do the same. We have the following two theorems that relate uniform convergence of series of functions to when you can integrate and differentiate them term by term. These are essentially just restatements of the previous theorems on sequences, but applied to series via the sequence of partial sums.

Theorem 5.8 Let $u_k(x) : I \to \mathbb{R}$ be a sequence of integrable functions on some interval $I = [a, b]$ and suppose

$$\sum_k u_k(x)$$

converges uniformly on I. Then the sum

$$f(x) = \sum_{k=1}^{\infty} u_k(x)$$

is integrable on $[a, b]$ and

$$\int_a^b f(x)\,dx = \int_a^b \left(\sum_{k=1}^{\infty} u_k(x)\right) dx = \sum_{k=1}^{\infty} \left(\int_a^b u_k(x)\,dx\right)$$

Theorem 5.9 Suppose the series

$$\sum_{k=1}^{\infty} u_k(x) = f(x)$$

converges pointwise. If the differentiated series

$$\sum_{k=1}^{\infty} u_k'(x) = g(x)$$

is uniformly convergent, then the original series is uniformly convergent also and moreover $f'(x) = g(x)$.

Revisiting theorem 5.4, if we had a Fourier series of a piecewise C^n function with a jump discontinuity, even though the Fourier series converges pointwise from the convergence theorem, and even though we can integrate the series term by term, we could not hope to get uniform convergence on any interval containing the discontinuity. We would however be able to get uniform convergence on any closed interval away from the discontinuity.

Theorem 5.10 **Weierstrass M-test**
Suppose you have a sequence of functions $u_k(x) : I \to \mathbb{R}$ and suppose that for each k the function is bounded on I

$$|u_k(x)| \le m_k \quad \text{for all } x \in I,$$

where $m_k > 0$ is a positive constant. If the series

$$\sum_{k=1}^{\infty} m_k$$

converges, then the function series

$$\sum_{k=1}^{\infty} u_k(x) = f(x)$$

converges uniformly and absolutely to a function $f(x)$ for all $x \in I$. In particular, if the $u_k(x)$ are all continuous then so is the sum $f(x)$.

What does this say about Fourier series? Well, the summands of a Fourier series of a real function f, written in complex form are

$$u_k(x) = c_k e^{ikx},$$

and since f is real, and $|e^{ikx}| = 1$ for real x, we have that the summands are bounded by $|c_n|$. Moreover, for real functions $c_{-n} = \overline{c_n}$ and $|z| = |\overline{z}|$, then we have that for real functions with a Fourier series,

$$f \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx},$$

if the series of the absolute value (norm) of the coefficients

$$\sum_{k=-\infty}^{\infty} |c_k| = 2\sum_{k=0}^{\infty} |c_k| < \infty$$

converges, then the series

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

converges uniformly to its limit $\tilde{f}(x)$ which is a continuous function, with the same Fourier coefficients.

Interestingly, the M-test guarantees that if $\sum |c_k|$ converges, then the Fourier series converges uniformly to a **continuous function**, which we denote by $\tilde{f}(x)$. However, this continuous limit function $\tilde{f}$ is not necessarily identical to the original function f that was used to compute the Fourier coefficients. The reason is that the integrals defining the coefficients c_k are insensitive to the values of f at finitely many points. Thus, if we alter a function at a finite set of points, the resulting Fourier coefficients remain unchanged, and hence the Fourier series is the same.

In short, a function f with jump discontinuities and a "corrected" continuous version $\tilde{f}$ can share exactly the same Fourier series. The M-test ensures that the series converges to this continuous version $\tilde{f}$.

Now, typically, we've seen Fourier series coefficients look something like

$$\frac{C_n}{k^\alpha}$$

where C_n is a constant, or oscillating constants or something of that ilk. The important part is that $|C_n| < C$, a constant, for all n (the formal name for this is that c_n be $\mathcal{O}(1)$ or uniformly bounded by a constant). Then the p-series test says that if $\alpha > 1$, we have convergence of the series

$$\sum_{k=1}^{\infty} \frac{C}{k^\alpha}.$$

This means that by the Weierstrass M-test we must have uniform convergence and by the uniform convergence theorem, we must have a continuous limit. This has some striking implications.

First, starting with a Fourier series

$$f \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx},$$

where

$$|c_k| < \frac{C}{k^\alpha}$$

and differentiating termwise we have that the coefficients of the differentiated series will be kc_k, so

$$|kc_k| < \frac{Ck}{k^\alpha} = \frac{C}{k^{\alpha-1}}.$$

So in particular, if $\alpha > 2$ we have uniform convergence of the series of derivatives, so our original Fourier series was differentiable and we've got a formula for the derivative of the Fourier series. Generalising this we have

Theorem 5.11 If the Fourier coefficients satisfy

$$|c_k| \leq \frac{M}{k^\alpha}$$

for some $\alpha > n+1$, then the Fourier series converges uniformly to an n-times continuously differentiable 2π-periodic function.

This is really kind of striking — it says that the decay rate of Fourier series coefficients can tell you how smooth the limiting function is. In particular if you decay like $\frac{1}{k^2}$, you have a continuous function, almost differentiable (piecewise C^1 in fact), while better than $\frac{1}{k^{2+\alpha}}$ for any $\alpha > 0$ means you have a continuously differentiable (C^1) function. If your Fourier coefficients decay faster than any polynomial rate (say at an exponential rate for example), then the Fourier series will converge to an infinitely differentiable function.

Informally, in the other direction, we get one factor of $\frac{1}{k}$ for each derivative that is continuous (as a periodic function) starting with the 0^{th} one and ending with the first one that has a jump. We can see this by integrating by parts. Suppose that $f(x)$ is a continuous function with piecewise first derivative function then

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx = \left.\frac{f(x)\cos(kx)}{k\pi}\right|_{x=-\pi}^{\pi} - \frac{1}{\pi k}\int_{-\pi}^{\pi} f'(x)\cos(kx)\,dx$$

since f is continuous and periodical, $f(\pi) = f(-\pi)$

$$b_k = -\frac{1}{\pi k}\int_{-\pi}^{\pi} f'(x)\cos(kx)\,dx$$

but if f is piecewise C^1, then $|f'|$ has a bound, say M, so we can bound the last integral and we have that

$$|b_k| \leq \frac{C}{k}.$$

A similar argument holds for the a_k's. This can be seen in the Fourier series of the absolute value function.

Lemma 1 **Riemann-Lebesgue lemma**

Let $f(x)$ be a piecewise continuous function on $[-\pi, \pi]$, then its Fourier coefficients decay to zero as $k \to \infty$. That is,

$$\lim_{N\to\infty} a_N = \lim_{N\to\infty} \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(Nx)\,dx = 0$$

$$\lim_{N\to\infty} b_N = \lim_{N\to\infty} \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(Nx)\,dx = 0$$

5.6.1 Proof of the convergence theorem

Now we will prove the Convergence Theorem. We want to show that for a piecewise C^1 function f that is also 2π periodic we have that the limit of the partial sums of the Fourier series is

$$\lim_{n\to\infty} s_n(x) = \frac{f(x^+)+f(x^-)}{2} \tag{5.2}$$

We start by writing out the complex form of the partial sums of the Fourier series of f and substituting the formula for the coefficients

$$\begin{aligned} s_n(x) &= \sum_{k=-n}^{n} c_k e^{ikx} = \sum_{k=-n}^{n} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iky}\,dy\right)e^{ikx} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\sum_{k=-n}^{n} e^{ik(x-y)}\,dy. \end{aligned} \tag{5.3}$$

Now we want to look at this term

$$\sum_{k=-n}^{n} e^{ikt} = e^{-int}+\cdots+e^{-it}+1+e^{it}+\cdots+e^{int} = e^{-int}(1+e^{it}+\cdots+e^{i(2n)t})$$

where we set $t = x-y$ for simplicity. The key observation is that this is a (partial) geometric series so we can use the form of the partial sums

$$\sum_{k=0}^{m} ar^k = a+ar+ar^2+\cdots+ar^m = a\left(\frac{r^{m+1}-1}{r-1}\right).$$

With $m = 2n$ and $a = e^{-int}$ and $r = e^{it}$. We thus have

$$\begin{aligned} \sum_{k=-n}^{n} e^{ikt} &= \frac{e^{-int}(e^{i(2n+1)t}-1)}{e^{it}-1} = \frac{e^{i(n+1)t}-e^{-int}}{e^{it}-1} \\ &= \frac{e^{-it/2}(e^{i(n+1/2)t}-e^{-int})}{e^{-it/2}(e^{it/2}-e^{-it/2})} = \frac{e^{i(n+1/2)t}-e^{-i(n+1/2)t}}{e^{it/2}-e^{-it/2}} \\ &= \frac{\sin\big((n+1/2)t\big)}{\sin(t/2)}. \end{aligned} \tag{5.4}$$

Lemma 2 We have the following:

1. $\dfrac{\sin((n+1/2)t)}{\sin(t/2)} = e^{-int}+\cdots+e^{-it}+1+e^{it}+\cdots+e^{int}$ is 2π-periodic.
2. $\dfrac{1}{2\pi}\int_{-\pi}^{\pi}\dfrac{\sin((n+1/2)t)}{\sin(t/2)}\,dt = 1.$
3. $\dfrac{\sin((n+1/2)t)}{\sin(t/2)} = 1+2(\cos t+\cos 2t+\cdots+\cos nt).$

Now substituting eq. (5.4) into eq. (5.3), we have

$$s_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \, \frac{\sin\big((n+\frac{1}{2})(x-y)\big)}{\sin\big(\frac{1}{2}(x-y)\big)} \, dy. \tag{5.5}$$

Now let $\hat{y} = y - x$, so $\pm\pi \to \pm\pi - x$ and $d\hat{y} = dy$. Substituting in and cancelling the negatives of the trigonometric functions we have

$$s_n(x) = \frac{1}{2\pi} \int_{-\pi-x}^{\pi-x} f(\hat{y}+x) \, \frac{\sin\big((n+\frac{1}{2})\hat{y}\big)}{\sin(\hat{y}/2)} \, d\hat{y} \overset{\text{periodical}}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\hat{y}+x) \, \frac{\sin\big((n+\frac{1}{2})\hat{y}\big)}{\sin(\hat{y}/2)} \, d\hat{y}.$$

Thus to prove eq. (5.2), it suffices to show

$$\lim_{n\to\infty} \frac{1}{\pi} \int_0^{\pi} f(y+x) \, \frac{\sin((n+1/2)y)}{\sin(y/2)} \, dy = f(x^{\pm}). \tag{5.6}$$

The proofs of the formulas are identical, so we'll show the one with the $+$. We have

$$\frac{1}{\pi} \int_0^{\pi} \frac{\sin((n+1/2)y)}{\sin(y/2)} \, dy = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin((n+1/2)y)}{\sin(y/2)} \, dy = 1.$$

Using (2) in the lemma, if we multiply this integrand by $f(x^+)$ and take the difference between it and the left-hand side of eq. (5.6), we want to show

$$\lim_{n\to\infty} \frac{1}{\pi} \int_0^{\pi} \frac{f(x+y) - f(x^+)}{\sin(y/2)} \sin\big((n+1/2)y\big) \, dy = 0. \tag{5.7}$$

We claim that for each fixed value of x the function

$$g(y) = \frac{f(x+y) - f(x^+)}{\sin(y/2)}$$

is piecewise continuous for all $0 \le y \le \pi$. The only potential problem is at $y = 0$, but here we can use the one-sided L'Hopital's rule:

$$\lim_{y\to 0^+} g(y) = \lim_{y\to 0^+} \frac{f(x+y) - f(x^+)}{\sin(y/2)} = \lim_{y\to 0^+} \frac{f'(x+y)}{\frac{1}{2}\cos(y/2)} = 2f'(x^+).$$

Consequently, eq. (5.7) (and hence eq. (5.2)) will be established if we can show that

$$\lim_{n\to\infty} \frac{1}{\pi} \int_0^{\pi} g(y) \, \sin\big((n+1/2)y\big) \, dy = 0.$$

Expanding using the trigonometric formula for sums we have

$$\frac{1}{\pi} \int_0^{\pi} g(y) \sin\big((n+1/2)y\big) \, dy = \frac{1}{\pi} \left(\int_0^{\pi} g(y) \sin(y/2) \cos(ny) \, dy + \int_0^{\pi} g(y) \cos(y/2) \sin(ny) \, dy \right).$$

But each of these terms vanishes because of the Riemann-Lebesgue Lemma. □