Financial Math

Sequences and Series

A sequence is a list of numbers that follow a definite pattern.

Arithmetic sequence: Each term is obtained by adding a constant to the previous term. These sequences are in the form: $a, a+d, a+2d, a+3d, ...$

Geometric sequence: Each term is obtained by multiplying the previous term by a constant. These sequences are in the form: $a, ar, ar^2, ar^3, ...$

A series is the sum of the terms of a sequence. It might be finite (adding finitely many terms of a sequence) or infinite (adding infinitely many terms of a sequence).

Arithmetic series: Sum of the terms of an arithmetic sequence
Geometric series: Sum of the terms of a geometric sequence

Arithmetic Series

The specific term is given by:

$$T_n = a + (n-1)d$$

Sum of the first n terms of the arithmetic sequence:

$$\begin{aligned} S_n &= \frac{n}{2}(2a + (n-1)d) \\ &= \frac{n}{2}(a + l) \end{aligned}$$

Geometric Series

The specific term is given by:

$$T_n = ar^{n-1}$$

Sum of the first n terms of the geometric sequence:

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}$$

Infinite Geometric Series

If $|r| < 1$, then the sum of the infinite sequence exists. The sum of the terms of the geometric sequence is given by (assuming $|r| < 1$:

$$S_n = \frac{a}{1-r}$$

Simple and Compound Interest

Simple Interest

Simple interest is a fixed percentage of the principal P_0 that is paid to an investor each year.
If the investor invests P_0 at an annual rate of $i\%$ then after t years she will receive:

$$I = P_0 \times i \times t$$

Conversely, give the value of the investment after t years (P_t), after t years will be:

$$P_0 = \frac{P_t}{1 + it}$$

Compound Interest

Compound interest pays interest on the principal plus interest on any interest accumulated in previous years.
The value after year t is given by:

$$P_t = P_0(1 + i)^t$$

Given the value after t years (P_t) and the interest i, one can find the principal (P_0):

$$P_0 = \frac{P_t}{(1 + i)^t}$$

Interest Compounded Several Times per Year

Interest may be compounded multiple times a year. Each time period is known as a conversion period. If the number of conversion periods per year is denoted by m and the interest is $i\%$ per year then $\frac{i}{m}$ is applied each conversion period. That is, in t years, the value of an initial investment of P_0 will be:

$$P_t = P_0 \left(1 + \frac{i}{m}\right)^{mt}$$

Continuous Compounding

Continuous compounds is when the number of conversion periods approach infinity.
This means that after 1 year, the value of the investment will be:

$$\lim_{m \to \infty} P_0 \left(1 + \frac{i}{m}\right)^{m} = P_0 e^i$$

This is because as n get larger, $\left(1 + \frac{i}{n}\right)^n$ approached e^i.
So, the value of an investment P_0 after t years, when compounded continuously at interest rate i is:

$$P_t = P_0 e^{it}$$

Integration

Integration is the reverse of differentiation. The integral of a function $f(x)$ would be the function $g(x)$ whose derivative is $f(x)$. That is, the integral of $f(x)$ is $g(x)$ that satisfies $g'(x) = f(x)$.

Integration is expressed with the symbol $\int$. In particular, $\int f(x)\ dx = g(x)$ means that the integral of $f(x)$ is $g(x)$.

The Power Rule of Integration

The power rule of integration is directly derived from the power rule of differentiation. The power rule simply states the following:

$$\int x^n\ dx = \frac{x^{n+1}}{n+1} + C$$

For any value of n, that is not equal to -1, we can find the integral of x^n with the power rule. If, $n = -1$, it results in $\ln(x) + C$.

Further Rules for Integration

Integral of a Constant: For any constant K, $\int K\ dx = Kx + C$.

Sum and Difference: Integral of the sum (or difference) of several functions is the sum (or difference) of the integrals of each function. That is, $\int (f(x) \pm g(x))\ dx = \int f(x)\ dx \pm \int g(x)\ dx$

Multiplication by a Constant: The integral of a function multiplied by a constant is the integral of the function multiplied by the constant. That is, $\int Kf(x)\ dx = K \int f(x)\ dx$.

Integral of the Natural Exponential: The integral of the exponential function e^x is itself (plus a constant). That is, $\int e^x\ dx = e^x + C$.

Integration by Substitution

Integration by substitution is derived from the chain rule of differentiation.

Example: Evaluate $\int (x^3 + 10)^3 2x\ dx$. We can set $u = x^2 + 10$, which would imply that $\frac{du}{dx} = 2x$, and hence $du = 2x\ dx$. Substituting u and du into the intial equation, we have:

$$\begin{aligned}\int (x^2 + 10)^3 2x\ dx &= \int u^3\ du \\ &= \frac{u^4}{4} + C \\ &= \frac{(x^2 + 10)^4}{4} + C\end{aligned}$$

Integration by Parts

Integration by parts is derived from the product rule of differentiation. Integration by parts is given by the following equation:

$$\int f(x)g(x)dx = f(x)\int g(x)\ dx - \int f'(x)(\int g(x)\ dx)\ dx$$

Hint: Let $g(x)$ be the function that can be integrated.

Example:

$$\begin{aligned}
\text{Evaluate: } &\int \frac{\ln(x)}{x^2}\ dx \\
\text{Let } f(x) = \ln(x) &\text{ and } g(x) = x^{-2} \\
\therefore \int \frac{\ln(x)}{x^2}\ dx &= \ln(x)\left(-\frac{1}{x}\right) - \int \frac{1}{x}\left(-\frac{1}{x}\right)\ dx \\
&= -\frac{\ln(x)}{x} + \int x^{-2}\ dx \\
&= -\frac{\ln(x)}{x} + \frac{x^{-1}}{-1} + C \\
&= -\frac{\ln(x)}{x} - \frac{1}{x} + C
\end{aligned}$$