PDF Transformation: $f(y) = f_x(y^{-1}) * \frac{d}{dy}|y^{-1}|$, y^{-1} is x given in terms of y

TOPIC 1: ESTIMATION

Methods for Generating Estimators

Method of Moments

Use the sample raw moments (from the data) to estimate the population raw moments

There are k parameters to estimate: $\theta_1, \dots, \theta_k$

The rth raw (population) moment is $\boxed{\mu'_r = E(X^r)}$ which is a function of the k parameters.

FN OF THE DATA ONLY

The rth raw (sample) moment is $\boxed{M'_r = \frac{1}{n}\sum_{i=1}^n X_i^r = \overline{X^r}}$ which is a function of the n variables

The sample moment can be used to estimate the population moment by solving the equations obtained by setting $\mu'_r = M'_r$; $r = 1,2,\dots,k$

Maximum Likelihood Estimator

Suppose $X_1, X_2, \dots, X_n$ have joint pdf/pf $f(x;\theta)$ (not necessarily i.i.d). The likelihood function is thought of as a function of θ. It is a random function as it depends on the random vector X:
$$\mathcal{L}(\theta; X) = f_X(x;\theta)|_{x=X}$$
Essentially, the likelihood function $\mathcal{L}$ is the joint density function written in terms of the rvs themselves.

If the X_i are i.i.d with common pdf/pf, then

$$\mathcal{L}(\theta; X) = \prod_{i=1}^n f(x_i; \theta)|_{x_i = X_i}$$

Then maximise $\mathcal{L}$ or log likelihood with respect to $\theta_1, \dots, \theta_k$ by logic or calculus (maximum turning point)

Properties of Estimators

1. Unbiasedness

An unbiased estimator is one which has expected value equal to the value of the parameter (a constant) we are trying to estimate. General result: the expected value of a function of X is equal to the function of the expected value of X if the function is LINEAR. $bias(T) = $ difference between $E(T)$ and $\tau(\theta)$

2. Variance - Choose the estimator with the smallest variance

3. Mean Squared Error

Let T be an estimator of $\tau(\theta)$. The Mean Squared Error (mse) of T is:
$$mse(T) = E[\{T - \tau(\theta)\}^2] = var(T) + [bias(T)]^2$$

4. Relative Efficiency

Define the relative efficiency of T_1 relative to T_2 in estimating $\tau(\theta)$ as the ratio:
$$RE = \frac{mse(T_2)}{mse(T_1)}$$
If RE is < 1, then T_2 is said to be a better estimator that T_1. <u>However, this may depend on θ or n.</u>

5. Consistency

A consistent estimator narrows in on the target value as n increases (bias and variance tends to zero). A biased estimator can be consistent but must have: $bias \to 0$ as $n \to \infty$.

TOPIC 2: SAMPLING DISTRIBUTIONS

Distributions Derived from the Normal

1. Square of a Standard Normal (and sum of independent chi-sqs):

If $X \sim N(0,1)$, then $Y = X^2 \sim \chi_1^2$, that if $X_1, X_2, \dots, X_n$ are i.i.d χ_1^2, then

$$Y = \sum_{i=1}^{n} X_i^2 \sim \chi_n^2$$

And if $Y_1 \sim \chi_{v_1}^2$ independently of $Y_2 \sim \chi_{v_2}^2$, then $Y_1 + Y_2 = \chi_{v_1+v_2}^2$

$$[N(0,1)]^2 \sim \chi_1^2 \quad \rightarrow \quad \chi_v^2 \sim gamma\left(\frac{v}{2}, 2\right) \quad \rightarrow \quad \chi_m^2 + \chi_n^2 \sim \chi_{m+n}^2 \quad \rightarrow if\ rvs\ are\ independent$$

2. Students t Distribution

Let $X \sim N(0,1)$, independently of U $\sim \chi_v^2$. Let T be the rv given by

$$T = \frac{X}{\sqrt{U/v}}$$

Then we say that $T \sim t_v$ (T is distributed as Student's t with v degrees of freedom).

3. The F Distribution

Suppose $U \sim \chi_{v_1}^2$ and $V \sim \chi_{v_2}^2$ are independent. Define F by

$$F = \frac{U/v_1}{V/v_2}$$

Then $F \sim F_{v_1,v_2}$ (we say that F is distributed as F with v_1 and v_2 degrees of freedom).

4. The Beta Distribution

Let $Y \sim \Gamma(\alpha_1, \beta)$ independently of $X \sim \Gamma(\alpha_2, \beta)$. Let,

$$B = \frac{Y}{Y + X} \sim \frac{\Gamma(\alpha_1, \beta)}{\Gamma(\alpha_1, \beta) + \Gamma(\alpha_2, \beta)} \sim beta(\alpha_1, \alpha_2)$$

5. The Cauchy Distribution

$$T \sim \frac{N(0,1)}{|N(0,1)|} \sim Cauchy \quad \rightarrow \quad Z \sim \frac{N(0,1)}{N(0,1)} \sim Cauchy$$

Thus, Z has the same pdf as T, although it is obviously not equal to T. (It equals T with probability $\frac{1}{2}$ and $-T$ with prob. $\frac{1}{2}$.).

Sampling Distributions

Sample Mean: $\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$

Sample Variance: $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2 \quad \rightarrow \quad \frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$

Let $X_1, \ldots, X_n$ be i.i.d $N(\mu, \sigma^2)$. Then,

$$\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

1. Properties of $\overline{X}$ and S^2 when sampling from the normal

If X_i i.i.d $\sim N(\mu, \sigma^2)$, $\overline{X}$ and S^2 are independent random variables. This is true even though S^2 is a function of $\overline{X}$.

2. Distribution of the t- statistic

$H_0: \mu = \mu_0$

$$T = \frac{\overline{X} - \mu_0}{S/\sqrt{n}}$$

When H_0 is true,

$$T \sim t_{n-1} \sim \frac{N(0,1)}{\sqrt{\chi_{n-1}^2/(n-1)}}$$

3. Distribution of the F statistic

Suppose $X_{11}, X_{12}, \ldots, X_{1m}$ are i.i.d $N(\mu_1, \sigma_1^2)$ and $X_{21}, X_{22}, \ldots, X_{2n}$ are i.i.d $N(\mu_2, \sigma_2^2)$ and independent of the other.

$$H_0: \sigma_1^2 = \sigma_2^2 \quad \rightarrow \quad H_0: \frac{\sigma_1^2}{\sigma_2^2} = 1$$

If H_0 is true, there is one variance, σ^2. Thus,

$$F = \frac{S_1^2}{S_2^2} = \frac{S_1^2/\sigma^2}{S_2^2/\sigma^2} \sim \frac{\chi_{m-1}^2/(m-1)}{\chi_{n-1}^2/(n-1)} \sim F_{m-1,n-1}$$

TOPIC 3: INTERVAL ESTIMATION (pivotal quantities) – AYSMPTOTIC DISTRIBUTION OF THE MLE

1a. Symmetric Confidence Intervals (σ^2 known)

$\rightarrow$ the $100(1-\alpha)\%$ symmetric CI for θ is $\left(\overline{X} \pm z_{\alpha/2} \times \frac{\sigma}{\sqrt{n}}\right)$

1b. Non-Symmetric Confidence Intervals (σ^2 known)

$\rightarrow$ the $100(1-\alpha)\%$ CI for θ is $\left(-\infty, \ \overline{X} + z_{\alpha} \times \frac{\sigma}{\sqrt{n}}\right), Ha: \mu < \mu0$

$\rightarrow$ the $100(1-\alpha)\%$ CI for θ is $\left(\overline{X} - z_{\alpha} \times \frac{\sigma}{\sqrt{n}}, \ \infty\right), Ha: \mu > \mu0$

σ^2 unknown:

$\rightarrow$ the $100(1-\alpha)\%$ symmetric CI for θ is $\left(\overline{X} \pm t_{n-1,\alpha/2} \times \frac{S}{\sqrt{n}}\right)$