Contents

Individuals' utility maximisation

1. BC for young:

$$c_{yt} + \alpha_{t+1} = w_t$$

2. BC for old:

$$c_{ot+1} = r_{t+1}\alpha_{t+1} + \alpha_{t+1} = a_{t+1}(1 + r_{t+1})$$

   a. BC for old with **depreciation**: only principal depreciates; need to introduce δ[1]

$$c_{ot+1} = r_{t+1}\alpha_{t+1} + (1-\delta)\alpha_{t+1} = a_{t+1}[(1-\delta) + r_{t+1}]$$

3. BC combined:

$$c_{ot+1} = (w_t - c_{yt})(1 + r_{t+1})$$

$$c_{yt} + \frac{c_{ot+1}}{1 + r_{t+1}} = w_t$$

   a. This BC is now 'dynamic' – it incorporates two time periods
      i. Where $\frac{1}{1+r_{t+1}}$ is the discount factor

Step 1: individuals' utility maximisation	
1. $\max_{c_{yt}, c_{ot+1}, \lambda} \mathcal{L} = u(c_{yt}, c_{ot+1}) + \lambda\left[w_t - c_{yt} - \left(\frac{c_{ot+1}}{1+r_{t+1}}\right)\right]$	Maximise the Lagrangian
2. $\frac{\partial \mathcal{L}}{\partial c_{yt}} \to \frac{\partial \mathcal{L}}{\partial c_{t+1}} \Rightarrow \frac{\alpha}{1-\alpha}\frac{c_{ot+1}}{c_{yt}} = 1 + r_{t+1}$	Derive Euler equation, by dividing
3. $c_{ot+1} = \frac{1-\alpha}{\alpha}(1 + r_{t+1})c_{yt} \to \frac{\partial \mathcal{L}}{\partial \lambda}$ Solve for c_{yt}^* Usually $c_{yt}^* = \alpha w_t$	Rearrange Euler to $c_{ot+1}(c_{yt})$ and substitute into BC FOC to solve for optimal consumption when **young**
4. $\alpha_{t+1}^* = w_t - c_{yt}^* = (1-\alpha)w_t$	Solve for savings by using the BC for young
5. $c_{ot+1}^* = (1 + r_{t+1})\alpha_{t+1}^* = (1 + r_{t+1})(1-\alpha)w_t$	Solve for optimal consumption when old using BC for old and substituting in optimal savings

- Need to compare value of LHS (utility gain of spending \$1 today) and RHS (utility gain of spending \$1 tomorrow) of Euler equation
  - The side which is higher yields greater utility and will determine consumption behaviour
- Note α_{t+1}^* is independent of r_{t+1} when $u(\cdot)$ is a C-D function – otherwise, will depend on r_t
  - This is because the IE and SE exactly offset each other
    - SE: $\uparrow r_{t+1}$ provides more incentive to save when young, hence $\uparrow \alpha_{t+1}$, as can yield a greater return when old
    - IE: $\uparrow r_{t+1}$ provides more incentive to consume in both periods, as higher returns to savings α_{t+1}, hence $\uparrow c_{yt}$ and c_{ot+1}
      - Savings, and consumption whilst young are unchanged, although consumption whilst old will increase

Firms' profit maximisation

1. Constant returns to scale and competitive markets allow us to represent all firms with one representative firm production function:

$$Y_t = A_t K_t^{\beta} L_t^{1-\beta}$$

   a. Where K_t, L_t represent firm demand

Step 2: firms' utility maximisation	
1. $\max_{K_t, L_t} \Pi = A_t K_t^{\beta} L_t^{1-\beta} - r_t K_t - w_t L_t$	Unconstrained maximisation problem
2. $\frac{\partial \Pi}{\partial K_t} = 0 \to r_t = \beta A_t \left(\frac{K_t}{L_t}\right)^{\beta-1}$	Take FOCs and solve for r_t, w_t, keeping capital to labour ratio $k_t \equiv \frac{K_t}{L_t}$. Can see that prices paid r_t and

[1] Only need to introduce δ to one side (i.e. household side)

3. $\frac{\partial \Pi}{\partial L_t} = 0 \rightarrow w_t = (1-\beta)A_t\left(\frac{K_t}{L_t}\right)^{\beta}$	w_t are equal to their marginal products, and they are decreasing (increasing) in k_t
4. $\frac{r_t^* K_t}{Y_t} = \beta \leftrightarrow r_t^* K_t = \beta Y_t$ 5. $\frac{w_t^* L_t}{Y_t} = (1-\beta) \leftrightarrow w_t^* L_t = (1-\beta)Y_t$	Substitute r_t^*, w_t^* into factor share equations and show factor shares equal their exponents. Rearrange so can substitute into Π equation
6. $r_t^* K_t \rightarrow \Pi, w_t^* L_t \rightarrow \Pi$ where $\Pi^* = 0$	Substituting into Π equation shows that with perfect competition, firms make zero profits. If not competitive/not constant returns to scale, would need to model firm entry/exit

Aggregation

1. Total supply of labour (by young individuals): N
   a. Total demand of labour (by firms): L_t
2. Total supply of capital (by old individuals): $N\alpha_t$
   a. Total demand of capital (by firms): K_t

Equilibrium (all markets clear; transition equation)

1. Labour market clearing:

$$Demand = Supply$$
$$L_t = N$$

   a. With pop. Growth:

$$L_t = N_t$$

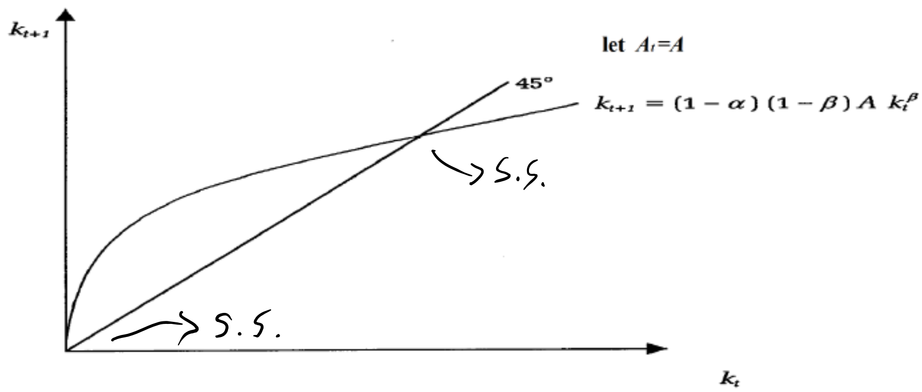
2. Capital market clearing:

$$Demand = Supply$$
$$K_t = N\alpha_t{}^2$$

   a. With pop. growth (as it is the previous generation which saves to provide capital in next period:

$$K_t = N_{t-1}\alpha_t$$

Step 3: derive transition equation	
1. $K_{t+1} = N\alpha_{t+1}$	Capital in $t+1$ is what the old generation invests
2. $k_{t+1} = \frac{K_{t+1}}{L_{t+1}} = \frac{K_{t+1}}{N_{t+1}} = \frac{N\alpha_{t+1}}{N_{t+1}} = \alpha_{t+1}$	Use labour and capital market clearing conditions
3. $k_{t+1} = \alpha_{t+1}^* = (1-\alpha)w_t$	Use α_{t+1}^* from consumer problem[3]
4. $k_{t+1} = (1-\alpha)w_t^* = (1-\alpha)(1-\beta)A_t k_t^{\beta}$	Use w_t^* from firm problem



Can see how capital stock grows over time.

45[0] line is where $k_{t+1} = k_t$ or $\Delta k_t = 0$, i.e., steady state.

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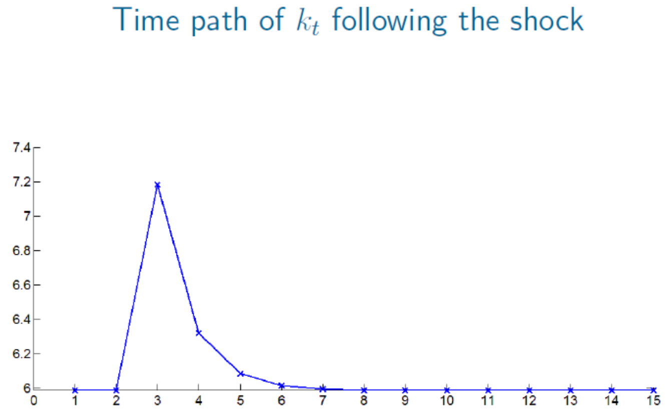
[2] α_t instead of α_{t+1} as it is the previous generation which supplies the capital

[3] Can use α_{t+1} from BC_y if $\alpha = 0$

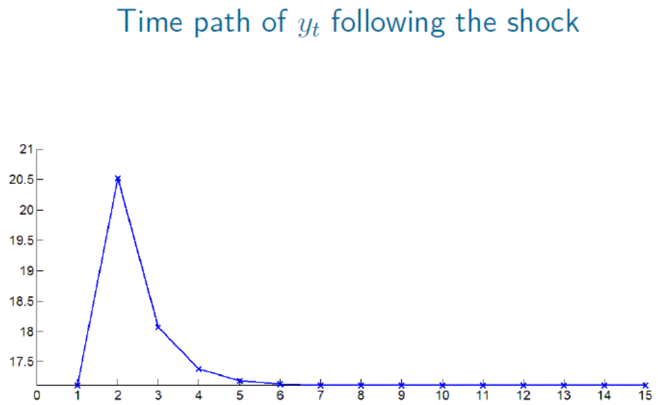


No effect on impact:

- It takes until the period following the shock for k_t to change. I.e., $k_1 = k_2 = \bar{k}$, as capital is predetermined by savings of young in $t = 1$ and $L = N$ is constant

Persisting effects:

- $k_t \uparrow$ as firms demand more capital, as the increase in productivity A_2 has increased the MP_K
- When shock dissipates, $k_t \downarrow$ over time (not instantly)
    - Characteristic of a business cycle

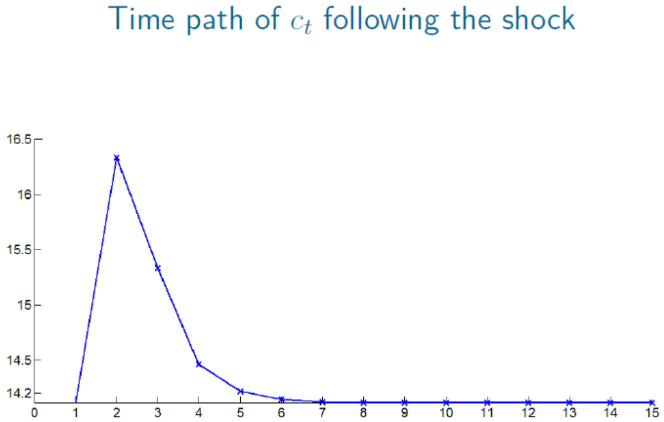


Effect on Impact:

- 20% increase, as there are no other changes to factor inputs other than the shock to A_2, as k_2 has not changed
    - CRS: the shock makes existing capital and labour inputs 20% more productive

Persisting effects:

- Same as k_t, as $y_t(k_t)$
    - As increased wages has increased the savings of workers

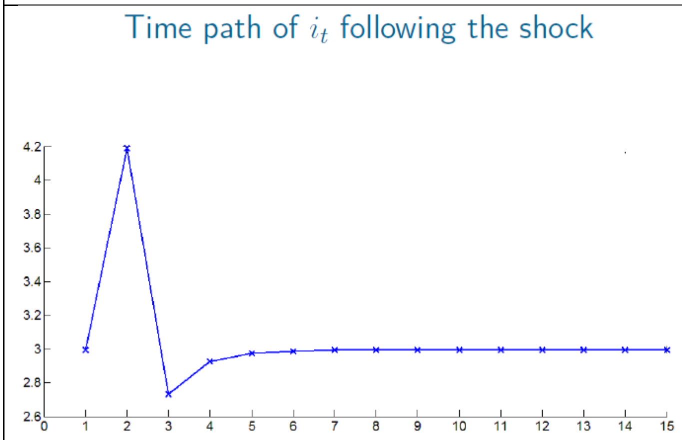


Effect on impact:

- Instant effect as pro-cyclical with y_t, although not as big, as less volatile

Persisting effects:

- As we use a C-D function, households **smooth consumption**, hence, there are greater persisting effects than output
    - The increased wage means the young generation in $t = 2$ save more yielding additional consumption in $t = 3$, as households' utility functions represent a **preference to consume in both generations**, hence they will save some



Effects on impact:

- Increases with r_t as households are willing to save more as they use rational expectations to expect higher capital returns when older

Persisting effects:

- At $t = 3$ after the shock, investment falls below steady state value, due to output falling more quickly than consumption through the relationship $I_t = Y_t - C_t$
- Following $t = 3$, investment gradually reaches its steady state value

$$\pi_{t+1} = \frac{P_{t+1}}{P_t} - 1 = \frac{M_{t+1}}{M_t} - 1$$

Lecture 17: *Is money neutral?*

Transition equation with money

Deriving transition equation with money assuming $\alpha = 0$	
$K_{t+1} = N\alpha_{t+1}$	Capital in $t+1$ is what the old generation invests
$k_{t+1} = \alpha_{t+1}$	Use labour and capital market clearing conditions
$k_{t+1} = w_t - \frac{m_{t+1}}{P_t}$ $= w_t - \gamma(i_{t+1})w_t$ $= [1 - \gamma(i_{t+1})]w_t$	Use α_{t+1} from BC_y Use $\frac{m_{t+1}}{P_t} = \gamma(i_{t+1})w_t$ Factorise
$k_{t+1} = [1 - \gamma(i_{t+1})](1-\beta)A_t k_t^{\beta}$	Use w_t^* from firm problem as unchanged

- This equation can be expanded further using:
  - Money demand function $\gamma(i_{t+1}) \equiv \theta\left(1 + \frac{1}{i_{t+1}}\right)$;
  - $i_{t+1} = \frac{P_{t+1}}{P_t}(1 + r_{t+1}) - 1$;
  - Price level and demand relation $P_t = \frac{M_t}{N(w_t - k_{t+1})}$
- To yield:

$$k_{t+1} = \left[1 - \theta\left(\frac{1}{\frac{M_{t+1}}{M_t}\frac{A(1-\beta)k_t^{\beta} - k_{t+1}}{A(1-\beta)k_{t+1}^{\beta} - k_{t+2}}\left(1 + A\beta k_{t+1}^{\beta-1}\right) - 1}\right)\right]A(1-\beta)k_t^{\beta}$$

  - With constant money supply $M_t = \bar{M}$ and all $k_t = \bar{k}$:

$$\bar{k} = \left[\frac{A(1-\theta)\beta(1-\beta)}{\theta + \beta(1-\theta)}\right]^{\frac{1}{(1-\beta)}}$$

  - Can see $\bar{k}$ does not depend on money constant money supply $\bar{M}$, implying **L-R money neutrality** In L-R:
    - $k,\ Y, r, w$ are all constant
    - Inflation rate is zero as equals rate of money supply growth, which is constant and equal to zero: $\pi_{t+1} = \frac{P_{t+1}}{P_t} - 1 = \frac{M_{t+1}}{M_t} - 1 = 0$
      - $i_t = r_t$ through fisher relation, as $\pi = 0$
  - And steady state price level $\bar{P} = \frac{\bar{M}}{N(\bar{w} - \bar{k})}$

Is money neutral in the long run

1. Yes, as can see $\bar{k}$ does not depend on money and;
2. Through price level and demand relation $P_t = \frac{M_t}{N(w_t - k_{t+1})}$, can see a change in the money supply increases the price level (denominator of real variables) by the same percentage

Is money neutral in the short run

1. One time increase in money supply, i.e., $M_t = \bar{M} + \Delta M,\ \ t \geq 2$
2. As mentioned above, no real L-R effects as the change in money supply increases the price level by the same percentage
3. S-R effects depend on what the newly printed money in $t = 2$, ΔM is used for: gov. buys goods in $t = 2$ and (1) transfers them to old (2) transfers them to young or (3) consumes the goods themselves: