MAST20026 Real Analysis - Lecture Notes

Topic 1 - Logic and Proof

Logic is about how to infer true things from known or assumed things. **Propositional logic** is about two valued variables and two valued functions of those variables. These two values will be called 'true' and 'false' or sometimes 0 and 1.

A **statement** is a sentence or expression that is either true or false. The statement takes on the role of a logical variable (the variables are true or false). One will generally use lower case letters *p, q, r* to represent statements.

One can combine statements to give new statements. Such statements are called **compound statements** (or logical formulae). They are constructed from primitive statements (that is, logical variables), connectives and parentheses (to remove ambiguity). Generally, one uses upper case letters A, B, C to denote compound statements.

If p and q are statements, the **disjunction** of p and q is the statement 'p or q' and is denoted by $p \vee q$ It is true if one or both of p and q are true, and it is false if both p and q are false. The truth table is given by:

P	Q	P ∨ Q
T	T	T
T	F	T
F	T	T
F	F	F

The disjunction is an inclusive use of 'or', that is; it allows for the possibility that both statements are true.

If p is a statement, the **negation** of p is the statement 'not p' and is denoted by $\sim p$. Its logical value is the opposite of p. Its truth table is given by:

p	$\sim p$
T	F
F	T

If p and q are statements, the **conjunction** of p and q is the statement 'p and q', denoted by $p \wedge q$. Conjunction accords with the natural language use of the word 'and'. Its truth table is given by:

p	q	p∧q
T	T	T
T	F	F
F	T	F
F	F	F

A statement of the form 'if p then q' is called an **implication** (or a conditional statement). The statements p and q have truth values that are independent of each other. The truth table is given by:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

The statement p is called the **antecedent** and the statement q is called the **consequent**.

A statement of the form 'p if and only if q' is called an **equivalence** (or a biconditional statement). Its truth table is given by:

p	q	p↔q
T	T	T
T	F	F
F	T	F
F	F	T

Statements constructed from several primitives are called compound statements. A compound statement is also either true or false. Another way to look at it is, if one combines statements with connectives in a valid way then one has a compound statement which is either true or false, and its truth value depends on the truth values of the statements comprising it.

Statements which have the same truth table are said to be **logically equivalent**. That is, p and q are logically equivalent if the biconditional statement is a **tautology**. That is, if the truth table of a compound statement is true for all values then it is called a tautology. The following are examples of logically equivalent statements:

1. $p \wedge q \equiv q \wedge p$
2. $p \vee q \equiv q \vee p$
3. $\sim (p \wedge q) \equiv (\sim p) \vee (\sim q)$
4. $\sim (p \vee q) \equiv (\sim p) \wedge (\sim q)$

Mathematical theories have several ingredients that propositional logic cannot easily represent (for example, sets, functions and mathematical operations). We use mathematical variables $x, y, z, \ldots$ to represent elements of the set of objects U (called the **universal set**). Formulae which depend on (free) variables are called **conditions** (or predicates).

If x is replaced by a particular value (or element of a set), then the formulae become statements. There are (at least) two ways to convert formulae containing mathematical variables into logical statements. The first is substitution; pick any element of U and substitute it into the formula. Another way is to use quantifiers. For example, the phrases 'for all' and 'there exists' are quantifiers, e.g.

- for all $x \in \mathbb{Z}, x^2 - 3x + 2 = 0$

- there exists an $x \in \mathbb{Z}$ such that $x > 0$

The phrase 'for all' is called the **universal quantifier** and is written as ∀. The phrase 'there exists' is called the **existential quantifier** and is written as ∃.

If Q is the statement ∀*x* in D s.t. *p(x)*, where D is a set and *p(x)* is a condition on *x*, then Q is true only if *p(x)* is be true for every member of D (this might be difficult to show). However, if at least one instance is false, then Q can be disproved if at least one false can be found. This is called a **counterexample**.

Similarly, if R is the statement ∃*x* in D s.t. *p(x)*, then R is true if there is one true instance of *p(x)*. So R can be proved true if at least one primitive statement is true; this is called an **example**. Hence, in general, one would disprove a universal statement with a counterexample and prove an existential statement with an example.

Since quantified conditions are statements, their negation is well defined. For the negation of the universal quantifier:

$$\sim(\forall x \in D \quad p(x)) \equiv \exists x \in D \quad \sim p(x)$$

Similarly, the negation of the existential quantifier is given by:

$$\sim(\exists x \in D \quad p(x)) \equiv \forall x \in D \quad \sim p(x)$$

If the truth table of a compound statement is false for all values of statements, then it is called a **contradiction**. Also note that the **contrapositive** of the conditional statement is given by:

$$p \Rightarrow q \equiv (\sim q) \Rightarrow (\sim p)$$

The converse of a conditional statement is given by:

$p \Rightarrow q$ is the statement $q \Rightarrow p$.

Note that a conditional statement is not logically equivalent to its converse. The following is also a very useful form of the biconditional. If *p* and *q* are statements, then:

$$[p \Leftrightarrow q] \equiv [(p \Rightarrow q) \wedge (q \Rightarrow p)]$$

The idea of inference is that by assuming some statements are true, and thus restricting rows of the truth table, rules of inference enable us conclude other statements are true. That is, from true statements we infer new true statements. Rules of inference are theorems in logic that allow us to infer that a given statement is true if a set of statements are all true.

The big advantage of inference rules is that, once they are proved, they can be used to infer new true statements without directly using truth tables. Thus, we get:

$$\{p, p \Rightarrow q\} \models q$$

This is a famous rule of inference called **Modus Ponens** or the Law of Detachment. It is valid as long as the first two statements are assumed (or known) to be true. If the set of statements are never all true then it is called an **inference contradiction**. That is, there is no row of the truth table where all the statements are true. This is weaker than the propositional form of contradiction as it does not require a whole column to be false.

There are many classical rules of inference. There are four primary rules of inference of propositional logic, which are:

1. **Contraction Rule** (infer A from $A \vee A$)
2. **Expansion Rule** (infer $B \vee A$ from A)
3. **Associative Rule** (infer $(A \vee B) \vee C$ from $A \vee (B \vee C)$)
4. **Cut Rule** (infer $B \vee C$ from $(A \vee B) \wedge (\sim A \vee C)$)

Modus Ponens is a theorem in logic, whose proof gives us a clue as to how to write proofs in mathematics. The pattern of the Modus Ponens proof is:

1. Start with some statements assumed true.
2. Infer a sequence of true statements that logically follow (i.e. using rules of inference, logical equivalence, tautologies, etc) from previous statements.
3. Arrive at a true statement.

In mathematics, proofs are written in an abbreviated form which summarises the important statements in such a way that is clear to the reader that true statements have been inferred from previous true statements. 'Clear to the reader' means the reader accepts that a full first order logic proof could be written (i.e. where every inferred step is justified by a rule of inference) but, in the interests of brevity and clarity, has not been so written.

True statements can be used as steps in a proof without inference come in various flavours, for example:

1. **Premises** - statements assumed true for the purposes of a proof.
2. **Axioms** - very simple statements assumed true for the purposes of building an area of mathematics.
3. Tautologies - always true.
4. Theorems - already proved true.
5. Definitions.

Thus, a proof is a finite sequence of statements such that they are either known or assumed true, or can be inferred from a known or assumed true statement (i.e. a previous statement). The last statement is usually called the **conclusion**.