The **pointwise convergence theorem** for Fourier series states that if a function f on [-L, L] is bounded and has finitely many critical points (including discontinuities), then:

- The Fourier series of f exists, i.e. the integrals defining the Fourier coefficients all converge.
- This Fourier series converges to the value F(x) of the periodic extension F of f, for every x in the domain where F is continuous.
- For any x values where F is discontinuous, the Fourier series instead converges to:

$$\frac{1}{2}\left(\lim_{w\to x^-} F(w) + \lim_{w\to x^+} F(w)\right)$$

A standard application of the convergence theorem is to evaluate sums. From the previous example, one can substitute x = L/2 which gives:

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} = \frac{\pi}{4}$$

The convergence theorem also helps one understand why the convergence rate of the Fourier coefficients seems to be tied to the smoothness of the periodic extension. Note that Fourier series can sometimes be term by term differentiated and integrated.

The maximum error at the discontinuity points does not converge to 0, but rather to about 8.95% of the gap, as the order of the Fourier approximation goes to infinity. This is known as the **Gibbs phenomenon**. It is problematic in many applications including compression algorithms for discontinuous data series. One fix is to replace the Fourier eigenfunctions by eigenfunctions that are discontinuous.

Recall that this is known as non-uniform convergence. Hence, if f' is piecewise continuous on [-L, L] and the periodic extension of f is continuous on the domain, then the Fourier series of f exists and it converges uniformly to F on all numbers, known as the **uniform convergence theorem**.

Recall that the BVPs were originally defined with boundaries at x = 0 and x = L for convenience. One common trick is to extend a function on [0, L] to an odd or even function on [-L, L] and assume periodic boundary conditions at these boundaries. Odd extensions will have Fourier series with no cos terms; even extensions will have Fourier series with only cos terms. Usually one wants a Fourier series in terms of the eigenfunctions appropriate for the boundary conditions.

Topic 5 - Partial Differential Equations

Partial differential equations (PDEs) are differential equations in which at least one of the differential operators is a partial derivative. For example:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + u = 1 + \sin(xy)$$

The unknown function u is therefore a function of several variables.

A PDE is linear if it says that a linear combination of partial derivatives of the unknown function, whose coefficients may be functions of the dependent variables, is equal to a function of the dependent variables. The order of a PDE is the order of the highest derivative. Note that general solutions of PDEs can involve unknown functions, for example:

$$\partial_t f(\boldsymbol{x}, t) = 0 \;\Rightarrow\; f(\boldsymbol{x}, t) = g(\boldsymbol{x})$$

Initial conditions for PDEs therefore constrain the unknown function at a given time to be a specified function. For example:

$$\begin{aligned} &\dot{u} = \alpha \nabla^2 u, \qquad && u(\boldsymbol{x}, 0) = f(\boldsymbol{x}); && \\ &\ddot{u} = c^2 \nabla^2 u, \qquad && u(\boldsymbol{x}, 0) = f(\boldsymbol{x}), \qquad && \partial_t u(\boldsymbol{x}, 0) = g(\boldsymbol{x}) \end{aligned}$$

When the PDE is order-*n*, it is natural to impose *n* conditions of this type at the initial value. Boundary conditions usually refer to constraints imposed at particular points in space, but for all positive times. If D denotes some (bounded) spatial domain, then boundary conditions are imposed on the domain's boundary. The two most common types of boundary conditions are:

- **Dirichlet boundary conditions**, which correspond to completely specifying the boundary values (for all time):

$$u(\boldsymbol{x}, t) = f(\boldsymbol{x}, t), \quad \text{for all } x \in \partial D \text{ and all } t > 0$$

- **Neumann boundary conditions** which correspond to completely specifying the values of the (outwards) normal derivative at the boundary (for all time):

$$\partial_{\boldsymbol{n}} u(\boldsymbol{x}, t) = g(\boldsymbol{x}, t), \quad \text{for all } \boldsymbol{x} \in \partial D \text{ and all } t > 0$$

We call a PDE with suitable initial and boundary conditions an initial-boundary value problem (IBVP).

The one-dimensional homogeneous **heat equation** is first-order in time and second-order in space, given by:

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}$$

One can find (at least some) solutions by assuming that the unknown temperature function *u* may be factorised as *X(x)T(t)*. Then, by **separation of variables**:

$$\frac{1}{D} \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}$$

Since the LHS is a function of *t* and the RHS is a function of *x*, the only way that they can be equal is if both are equal to a constant. Thus:

$$T'(t) + \lambda D T(t) = 0 \qquad \text{and} \qquad X''(x) + \lambda X(x) = 0$$

To proceed, one needs initial and boundary conditions.