MATH1003

Integral Calculus and Modelling

1 Introduction

Basic Terminology

To integrate, by definition, is to bring together parts into a whole. The term is used for any process of working out the size of some quantity from information about its rate of change.

Any constants will cancel out when differentiating an equation. For example, both $g(x) = 2x + 5$ and $h(x) = 2x + 9$ have the same derivative, $f'(x) = 2$. If presented with the integration problem of $f'(x) = 2$, it is firstly required to state the integral, $f(x) = 2x$, and secondly to include the constant value of C. Therefore, $f(x) = 2x + C$, which encompasses $g(x)$ and $h(x)$ simultaneously.

Introduction to Infinite Sequences

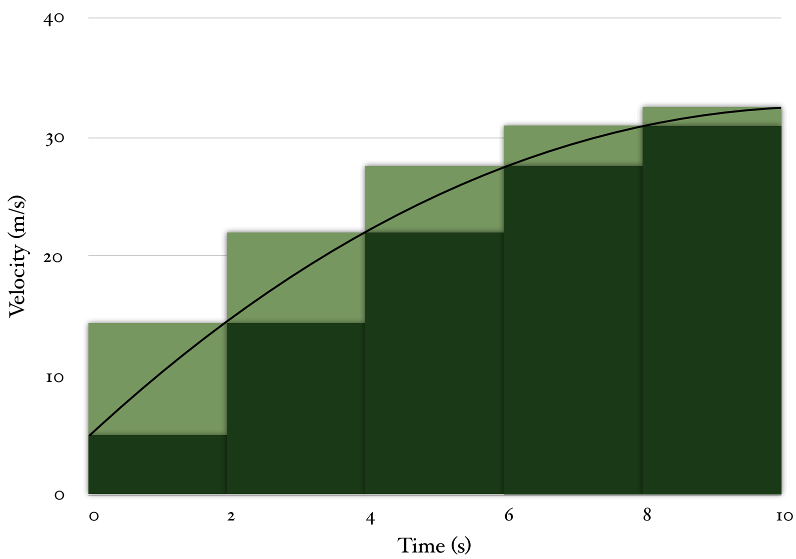
Infinite sequences may include arithmetic sequences such as the sequence of natural numbers, geometric sequences, the Fibonacci sequence and more. If a sequence has a finite limit, it is **convergent**. Otherwise, it is **divergent**. A **constant sequence** has a limit equal to the value of each term, for example, $a_n = 4$.

2 The Definite Integral: Definition

Velocity vs Distance Example

The more accurately a dependant variable, in this case velocity, is incrementally measured, the smaller the range will be between the minimum and maximum distances travelled. Initially, if we only measure velocity over a period of 10 seconds, at 0 seconds travelling 5 m/s and at 10 seconds travelling 32.5 m/s, only a wide approximation can be made. The minimum distance is 5 m/s x 10 s = 50 m and the maximum distance is 32.5 m/s x 10 m = 325 m. However, if time is measured every two seconds, the table and graph below will describe the upper and lower estimates for the distance travelled.

Time (s)	0	2	4	6	8	10
Velocity (m/s)	5	14.5	22	27.5	31	32.5



The **lower estimate** is the total area of the dark rectangles, (5 x 2) + (14.5 x 2) + (22 x 2) + (27.5 x 2) + (31 x 2) = 200 m. The **upper estimate** is then the total area of the dark and light rectangles, (14.5 x 2) + (22 x 2) + (27.5 x 2) + (31 x 2) + (32.5 x 2) = 255 m. The difference between these two estimates, the **error**, is the total area of the light rectangles, 255 m - 200 m = 55 m, equivalent to (32.5 - 5) x 2 = 55.

If the time increments become smaller, it becomes clear that this difference gets smaller and thus more accurate. Since the upper and lower estimates have a common limit as the increments approach zero, the limit being the curve, it may be concluded that the total distance travelled is the area under the curve.

Riemann Sums

A function is **continuous** when its graph is a single unbroken curve, free of holes, jumps or asymptotes. A non-negative continuous function $f(x)$ defined on an interval $[a, b]$ where $a \leq x \leq b$ can be divided into a **partition** of the interval, that is, $N \geq 1$ subintervals of equal length. The minimum and maximum values of $f(x)$ on each subinterval may then be drawn as rectangles to form respective lower and upper estimates. To express the sums in summation notation, $\Delta x = (b - a)/N$ is defined as the length of each subinterval, while m_i and M_i are the respective minimum and maximum values of $f(x)$ on the i-th subinterval.

Riemann Lower Sum
$$L_N = (m_1 \times \Delta x) + (m_2 \times \Delta x) + ... + (m_N \times \Delta x) = \sum_{i=1}^{N} m_i \times \Delta x$$

Riemann Upper Sum
$$U_N = (M_1 \times \Delta x) + (M_2 \times \Delta x) + ... + (M_N \times \Delta x) = \sum_{i=1}^{N} M_i \times \Delta x$$

As $\Delta x \to 0$ and $N \to \infty$, the upper and lower Riemann sums approach a common value, called the **definite integral** of $f(x)$ over the interval $[a, b]$. It is the unique number that satisfies the condition of lying between L_N and U_N for all $N \geq 1$. It is the integral of f over the interval.

The Definite Integral
$$\int_a^b f(x)\ dx$$

Non-Positive Functions

When dealing with functions that take on negative values, the summation process is the same except for a couple of factors. For the lower sum, any negative portions should still have its rectangles drawn below the curve. For the upper sum, any negative portions should also still have its rectangles drawn above the curve. Any rectangles below the axis count as negative in the summation. Therefore, both Riemann sums are equal to the sum of the the areas of all the rectangles above the axis, minus the sum of the areas of all the rectangles below the axis.

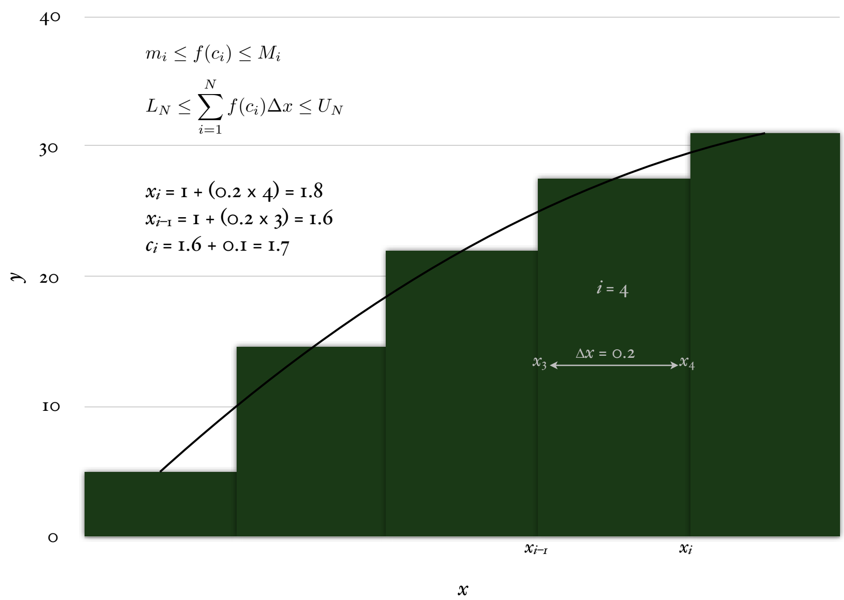
3 The Definite Integral: Properties

Practical Calculation of Riemann Sums

It is possible to choose points of reference for Riemann sums. c_i can be any point in the i-th subinterval and form a sum of areas of rectangles of width Δx and height $f(c_i)$, which is the general Riemann sum. c_i can be random, the midpoint, the left or the right endpoint of the subinterval.

General Riemann Sum
$$(f(c_1) \times \Delta x) + (f(c_2) \times \Delta x) + ... + (f(c_N) \times \Delta x) = \sum_{i=1}^{N} f(c_i) \times \Delta x$$

The graph below is an example of choosing the midpoint of each subinterval as c_i. If using a partition of $[1, 2]$ and $N = 5$ subintervals, each subinterval will have a length of $1/5 = 0.2 = \Delta x$.



Another example is using Riemann sums to estimate the integral ∫ sin *x dx* on the partition [1, 2] using 5 subintervals and the midpoint as c_i. The first step is to calculate the subinterval length, 0.2, then x_i and c_i. The second step is to work out the sum from the formula. The third step is to solve the sum. This is usually done in a spreadsheet but for a small number of subintervals, the solution may be shown in the table below. The integral can also be calculated exactly to check that the solution is a good estimate.

	Position: $x_i = 1 + 0.2 \times i$		Midpoint: $c_i = x_{i-1} + 0.1$			
i	x_{i-1}		c_i	$\sin c_i$		*Total*
1	x_0	1	1.05	0.867423225594017		4.69760813407437
2	x_1	1.1	1.15	0.912763940260521		
3	x_2	1.2	1.25	0.948984619355586		$\sum_{i=1}^{N} f(c_i) \times \Delta x = \sum_{i=1}^{5} \sin(c_i) \times 0.2$
4	x_3	1.3	1.35	0.975723357826659		
5	x_4	1.4	1.45	0.992712991037588		0.939521626814874

$$\int_1^2 \sin x \; dx = \int_1^2 \frac{d}{dx}(-\cos x) \; dx = (-\cos 2) - (-\cos 1) \approx 0.956449$$

Properties of the Definite Integral

I. When *m* and *M* are the minimum and maximum values of *f* on the interval [*a*, *b*], the integral always lies between them.

$$m \times (b - a) \leq \int_a^b f(x) \; dx \leq M \times (b - a)$$

II. If *c* is a constant, it may be taken from the right of the integral sign and placed to the left of the sign.

$$\int_a^b cf(x) \; dx = c \int_a^b f(x) \; dx$$

III. For functions *f* and *g* defined on the interval [*a*, *b*], the functions may be separated into an addition sequence.

$$\int_a^b (f(x) + g(x)) \; dx = \int_a^b f(x) \; dx + \int_a^b g(x) \; dx$$

IV. If *f* is defined on the interval [*a*, *c*], and *b* is a point between *a* and *c*, the integrals can be separated into parts of [*a*, *c*].

$$\int_a^c f(x) \; dx = \int_a^b f(x) \; dx + \int_b^c f(x) \; dx$$

Reversing the Direction of Integration

In the previous definition of the integral, the case was that $a \leq b$. However, if $a > b$, the subinterval length $\Delta x = (b - a)/N$ from the Riemann sum will become negative. Geometrically, the Riemann sum of areas of rectangles above the axis will consequently count as negative, and areas below the axis will count as positive. The sum simply changes sign according to Δx. It then follows that the fourth definite integral property holds true whatever the order of the numbers involved, which can rearranged and vertically flipped.

$$\int_a^b f(x) \; dx = -\int_b^a f(x) \; dx$$

The Fundamental Theorem of Calculus II

The **second part of the Fundamental Theorem** states to let *F*(*x*) be a function defined on an interval [*a*, *b*] of the real line, to suppose that the derivative of *F* is defined at each point *x* of the interval and to assume that the resulting function *F'*(*x*) is continuous. In this way, for a generic integral, there is a function *F*(*x*) with the property that *F'*(*x*) = *f* (*x*) on the interval [*a*, *b*]. *F*(*x*) is hence called the **antiderivative** of *f*(*x*).

Fundamental Theorem of Calculus II

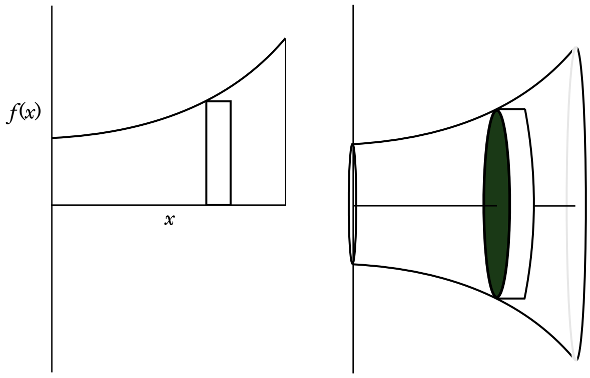
$$\int_a^b F'(x) \; dx = F(b) - F(a) = [F(x)]_a^b$$

4 The Definite Integral: Applications

Solids of Revolution – Volume by Disks

Using geometrical principles, it is feasible to approximate the volume of solids in the x-y plane. For continuous, positive functions defined on $[a, b]$, a solid is created by rotating the region bounded by $x = a$, $x = b$ and the graph of f about the horizontal x-axis $y = 0$. This is the **disk method**, as the interval can be partitioned into equal subintervals which create disks that form the upper and lower Riemann sums of the function. The method can also be used for y.

Disk Method



$$\sum_{i=1}^{N} \pi m_i^2 \times \Delta x \leq V \leq \sum_{i=1}^{N} \pi M_i^2 \times \Delta x$$

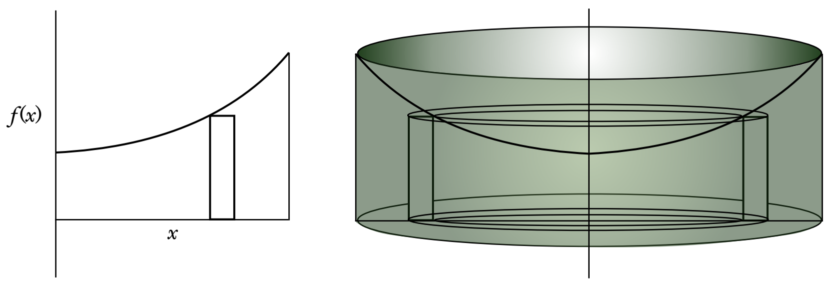
The definite integral is the only number that satisfies these inequalities. Intuitively, the Riemann sums, which are simply the sums $\Sigma \pi f(x)^2 \times \Delta x$, converge to the integral as N increases and as Δx approaches zero. $f(x)$ is the radius of the slice and $\pi f(x)^2$ gives the slice area, so taking slices all the way from a to b provides the exact volume of the solid.

$$V = \int_a^b \pi f(x)^2 \, dx$$

Solids of Revolution – Volume by Shells

The **shell method** is used for continuous, positive functions defined on $[a, b]$, the solid created by rotating the region bounded by $x = a$, $x = b$ and the graph of f about the vertical y-axis $x = 0$. This rotation generates a series of subintervals which form cylindrical shells of height $f(x)$ and thickness Δx. The method can also be used in reverse for x.

Shell Method



It is useful to visualise the cylindrical shell of the rotating subinterval inside the function cut and laid out flat. Its total length would then be the inner circumference $2\pi x$, where x is the distance from the origin to the inner barrier of the shell. Multiplying the circumference by the height of the shell, $f(x)$, would yield the area of the inner cylinder without any thickness. Therefore, the Riemann sum approximation of the volume of each 3D subinterval portion would be $2\pi x \times f(x) \times \Delta x$. The total volume is found using the exact integral as Δx approaches 0.

$$V = 2\pi \int_a^b x f(x) \, dx$$