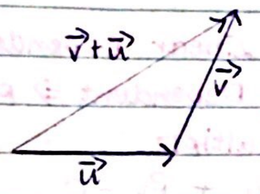
Maths 1002 :

Draw vectors : "Tail-to-Head"

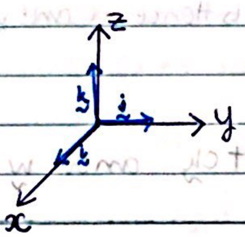


↳ vector v : $\underset{\sim}{v}$; $\vec{v}$
↳ length : $|\underset{\sim}{v}|$
↳ scalar : λ (is a real number)

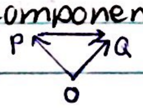
Unit vector : is a vector of length one: $\boxed{\hat{\underset{\sim}{v}} = \dfrac{\underset{\sim}{v}}{|\underset{\sim}{v}|}}$

Parallel vectors : $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are parallel $\Longleftrightarrow$ $\boxed{\underset{\sim}{v} = \lambda \underset{\sim}{w}}$

Cartesian form : $\underset{\sim}{v} = a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$ (a, b, c are components of $\underset{\sim}{v}$)



↳ add vectors ⇒ add components
↳ subtract vectors ⇒ subtract components
↳ scale vectors ⇒ scale components
↳ $\overrightarrow{PQ}$ = "Q" minus "P"



Length of a vector : If $\underset{\sim}{v} = a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$ then $\boxed{|\underset{\sim}{v}| = \sqrt{a^2 + b^2 + c^2}}$

Linear independence of two vectors :

* $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are linearly independent if the equation:
(not parallel)

$$\alpha \underset{\sim}{u} + \beta \underset{\sim}{v} = 0$$

holds only for the trivia solution:

$$\alpha = \beta = 0$$

* If there are non trivial solutions to this equation, we say $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are linearly dependent

E.g. 1) Show that vector $\underset{\sim}{a} = 3\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j}$ are LI (linearly independent)

2) Show that $\underset{\sim}{u} = 2\underset{\sim}{i} - 4\underset{\sim}{j}$ and $\underset{\sim}{v} = -3\underset{\sim}{i} + 6\underset{\sim}{j}$ are linearly dependent

Solutions:

Linear Independent

1) Suppose that $\alpha\underset{\sim}{a} + \beta\underset{\sim}{b} = 0$

$$\alpha(3\underset{\sim}{i} - \underset{\sim}{j}) + \beta(2\underset{\sim}{i} + \underset{\sim}{j}) = 0$$

$$3\alpha\underset{\sim}{i} - \alpha\underset{\sim}{j} + 2\beta\underset{\sim}{i} + \beta\underset{\sim}{j} = 0$$

$$\underset{\sim}{i}(\underbrace{3\alpha + 2\beta}_{=0}) + \underset{\sim}{j}(\underbrace{\beta - \alpha}_{=0}) = 0$$

Since $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are linearly independent,

$$3\alpha + 2\beta = 0$$
$$-\alpha + \beta = 0$$
$$\Rightarrow \alpha = \beta = 0$$

Hence $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are linearly independent.

Linear Dependent

2) Linearly Dependent $\Rightarrow$ parallel $\Rightarrow$ scalar multiple

$$\alpha\underset{\sim}{u} + \beta\underset{\sim}{v} = 0$$

$$\alpha(2\underset{\sim}{i} - 4\underset{\sim}{j}) + \beta(-3\underset{\sim}{i} + 6\underset{\sim}{j}) = 0$$

$$2\alpha\underset{\sim}{i} - 4\alpha\underset{\sim}{j} - 3\beta\underset{\sim}{i} + 6\beta\underset{\sim}{j} = 0$$

$$\underset{\sim}{i}(\underbrace{2\alpha - 3\beta}_{=0}) + \underset{\sim}{j}(\underbrace{6\beta - 4\alpha}_{=0}) = 0$$

$$2\alpha - 3\beta = 0$$
$$6\beta - 4\alpha = 0$$

$\Rightarrow \alpha = \frac{3}{2}\beta$ only condition!

↳ There are many solutions (e.g. $\alpha = 3$; $\beta = 2$) not just $\alpha = \beta = 0$

↳ Hence, $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are linearly dependent

(*) Dot product: (scalar quantity)

↳ Algebraic definition: If $\underset{\sim}{v} = a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$ and $\underset{\sim}{w} = d\underset{\sim}{i} + e\underset{\sim}{j} + f\underset{\sim}{k}$, then

$$\boxed{\underset{\sim}{v} \cdot \underset{\sim}{w} = ad + be + cf}$$

↳ Geometric definition: If $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are vectors and θ is angle between them, then:

$$\boxed{\underset{\sim}{v} \cdot \underset{\sim}{w} = |\underset{\sim}{v}|\,|\underset{\sim}{w}| \cos\theta}$$

↳ Dot product:

dot product $> 0 \Rightarrow \theta = 0^\circ$ or θ is acute

dot product $< 0 \Rightarrow \theta = 180^\circ$ or θ is obtuse

dot product $= 0 \Rightarrow$ two vectors are mutually perpendicular

↳ Cauchy - Schwarz Inequality:

$$\boxed{|\underset{\sim}{v} \cdot \underset{\sim}{w}| \leq |\underset{\sim}{v}|\,|\underset{\sim}{w}|}$$

↳ Dot product rules :

1) $\underset{\sim}{v} \cdot \underset{\sim}{w} = \underset{\sim}{w} \cdot \underset{\sim}{v}$
2) $(\underset{\sim}{u} + \underset{\sim}{v}) \cdot \underset{\sim}{w} = \underset{\sim}{u} \cdot \underset{\sim}{w} + \underset{\sim}{v} \cdot \underset{\sim}{w}$
3) $\underset{\sim}{v} \cdot \underset{\sim}{v} = |\underset{\sim}{v}|^2$, so $|\underset{\sim}{v}| = \sqrt{\underset{\sim}{v} \cdot \underset{\sim}{v}}$
4) $(\lambda \underset{\sim}{v}) \cdot \underset{\sim}{w} = \lambda (\underset{\sim}{v} \cdot \underset{\sim}{w}) = \underset{\sim}{v} \cdot (\lambda \underset{\sim}{w})$

5) Scalar component of $\underset{\sim}{u}$ in the direction of $\underset{\sim}{v}$:

$$\frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{v}|}$$

6) Vector projection of $\underset{\sim}{u}$ in the direction of $\underset{\sim}{v}$:

$$\frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{v}|^2} \underset{\sim}{v}$$

7) Vector component of $\underset{\sim}{u}$ orthogonal to $\underset{\sim}{v}$ is :
"Original minus vector projection"

$$\underset{\sim}{u} - \frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{v}|^2} \underset{\sim}{v}$$