2. FINANCIAL MATHEMATICS

Multiple future cash flows

Value Additivity: cash today cannot simply be added to cash tomorrow as CFs occur at different times

- Mixed stream – irregular (unstable) cash flows
- Therefore, you must **convert multiple CFs into a single equivalent cash flow** (CFs should be carried forward, accumulated, or back in time, discounted)

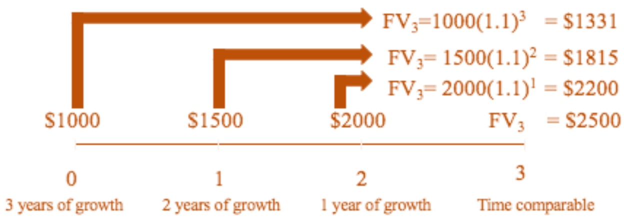
Future Value of a Mixed Stream

The approach to calculating the FV of a known mixed stream involves a 2 step process and requires **value additivity**

1. Calculate the FV of each future amount to be received at a *comparable point in time*

2. Sum all future values at a comparable point in time together to determine the future value of the known mixed stream

EXAMPLE: You deposit $1000 now, $1500 in 1 year, $2000 in 2 years and $2500 in 3 years in an account 10% interest per annum. How much do you have in the account at the end of the 3rd year?



1331 + 1815 + 2200 + 2500 = $7846

REMEMBER: add figures into calculator and **NPV** first, then **swap** for **FV**

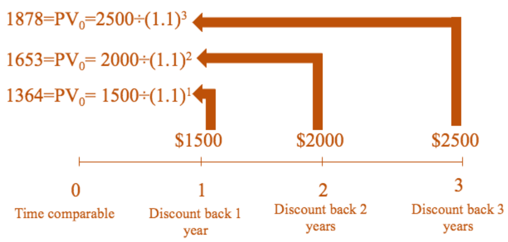
Present value of a Mixed Stream

Calculating PV also requires **value additivity**

1. Calculate the PV of each future amount to be received at a comparable point (normally t = 0)

2. Sum all PVs at a comparable point in time to determine the PV of the known mixed stream

EXAMPLE: You deposit $1500 in one year, $2000 in two years and $2500 in three years in an account paying 10% interest per annum. What is the PV of these cash flows?



1878.29 + 1652.89 + 1363.64 = 4894.82

REMEMBER: calculate **NPV**

Perpetuities

Perpetuity: a cash flow stream of equal amounts, equally spaced in time (*fixed regular payment, received at fixed points in time forever)*

- NOTE: as the formula *calculates PV at year 0,* if there is a CF in year 0, *add it onto the figure found* using the formula

$$PV_\infty = \frac{CF}{i}$$

where CF = the cash flow per period (i.e. the payment), i = the interest rate per period.

Note: The formula values cash flows ONE period BEFORE the first cash flow

EXAMPLE: government security promises to pay \$3 pa forever. If the interest rate is 10% pa and a payment of \$3 has *just been made,* how much is the security worth?

C = \$3, i = 10%

$$PV = \frac{CF}{i} = \frac{3}{0.1} = \$30$$

Note: as the \$3 has **already** been made, it is disregarded, and *not included*

EXAMPLE: government security promises to pay \$3 pa forever. If the interest rate is 10% pa and a payment of \$3 is *made tomorrow,* how much is the security worth?

$$PV = \frac{CF}{i} + C$$

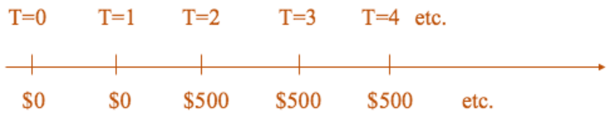
$$= \frac{3}{0.1} + 3 = \$33.00$$

Note: as the \$3 is made *in **year 0,*** it needs to be **added** to the calculation

EXAMPLE: Earn \$500 per year forever, with the first payment two years from today.

$$PV = \frac{C/i}{1+i} = \frac{\$500/0.08}{1.08} = \$5{,}787.04$$

Timeline



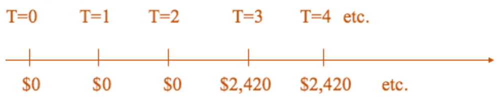
The first CF is in **year 2** – so the perpetuity formula will give the PV of *year 1*. This PV must be *discounted back to **year 0*** using the simple PV formula.

$PV = \frac{CF}{i}$ then $PV = \frac{FV}{(1+i)^n}$

EXAMPLE: \$2420 per year forever, with the first payment 3 years from today.

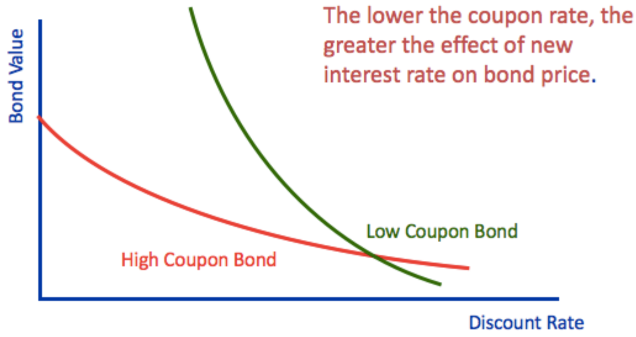
$$PV = \frac{C/i}{(1+i)^2} = \frac{\$2{,}420/0.08}{1.08^2} = \$25{,}934.50$$

Timeline



NOTE: ***Perpetuity Due →*** $PV = PMT + \frac{PMT}{i}$

2. **Level of coupon →** the *lower the coupon rate,* the **greater** the effect of the new interest rate through compounding (lower coupon bonds *more price sensitive;* firm pays you less so less immune to changes in the economy*)*



3. **Default risk →** *as default risk rises,* the value of a bond will **fall**

    - Risk for investors that the bond issuer may **default** on the bond's cash flow (coupon payments, repayment of bond at maturity)

    - The financial health of a company is assessed to determine the chance of *defaulting* → these assessments are summarised by **credit ratings,** which are provided by rating agencies such as S&P and Moody's

    - The **higher** the market's assessment of the probability of default, the **higher** the requires rate of return on the debt

Shares

Equity securities are a company's *certificates of ownership.*

- More than $1.14 trillion worth of public equity securities were outstanding in Australia in March 2010 – 44% of the adult population either own shares *directly* or indirectly (superannuation)

- Outstanding shares of a company are bought and sold among investors in the secondary market
    - Active secondary markets enable companies to sell their new debt/equity issues at lower funding costs & also provide investors with *fairly priced shares*

Share valuation

Value of share = ***PV of future cash flows*** *(but more complex due to uncertainty of CFs & no maturity)*

- *Expected cash flows* received from a share are **all future dividends** (consider dividend growth!)

Two approaches to share valuation – *dividend valuation* & *earnings per share (EPS):*

- Valuation based on dividends = dividends are discounted to a present value for share valuation
- Valuation based on earnings (EPS) = earnings are capitalised into a share value using a *price-earnings ratio*

Dividend based valuation

1) CONSTANT DIVIDEND VALUATION

The constant dividend assumption is the simplest – this is where dividends are **constant** (ie preference share) so the ***perpetuity formula*** can be applied → typically undertaken by mature firms who have constant profits

$$P_0 = \frac{D}{R}$$

EXAMPLE: Worley Ltd is expected to pay a constant annual dividend of $2.50 per share indefinitely. If the discount rate is 8% what is the share value?

$$P_0 = \frac{D}{R} = \frac{2.50}{0.08} = \$31.25$$

2) CONSTANT DIVIDEND GROWTH VALUATION

This is where dividends are expected to **grow at a constant rate *indefinitely*** (more realistic) → typically younger firms who are experiencing growth

$$P_0 = \frac{D_0(1+g)}{(R-g)}$$

where $g =$ expected growth rate in dividend per share

EXAMPLE: Coco Ltd has just paid an annual dividend of $0.30 per share which will grow at 5% indefinitely. RRoR is 8%, find share value

$$P_0 = \frac{D_0(1+g)}{(R-g)} = \frac{0.30(1+0.05)}{(0.08-0.05)} = \$10.50$$

[Note: D_0 is *current dividends;* if dividends are to be paid in 1 year, then *don't use $D_0(1 + g)$, just use **D_1**]*

3) VARIABLE DIVIDEND GROWTH VALUATION

This allows for *different growth rates* – dividends may grow at a high rate for a number of years, *but not indefinitely* (this valuation assumes dividends *will* grow constantly, some time in the future)

EXAMPLE: Dave Ltd just paid a $0.50 annual dividend. The RRoR on Dave's shares is 10%. Exceptional growth is forecasted for Dave Ltd for the next 3 years at 12% pa. After year 3, growth rate *(g)* will settle at 5% indefinitely. What is the value of Dave Ltd shares today?

$$P_0 = \frac{D_0(1+g')}{(1+R)} + \frac{D_0(1+g')^2}{(1+R)^2} + \frac{D_0(1+g')^3}{(1+R)^3} + \frac{1}{(1+R)^3} \times \frac{D_0(1+g')^3(1+g)}{(R-g)}$$

The first three terms account for the higher 12% growth (first three dividends), while the fourth term accounts for the all dividends thereafter (lower 5% growth).

$$P_0 = \frac{\$0.50(1.12)}{1.10} + \frac{\$0.50(1.12)^2}{(1.10)^2} + \frac{\$0.50(1.12)^3}{(1.10)^3} + \frac{1}{(1.10)^3}\frac{\$0.50(1.12)^3 1.05}{(0.10-0.05)}$$

$$= \$12.64$$

The value of Dave Ltd's shares is $12.64

Note: 1/(1 + R)3 discounts the dividends from year 3 onwards, as the PV found for those is PV3

Constant chain of replacement

To compare projects with different lives, we **replicate mutually exclusively projects** over time

The replacement chains are continued until both chains are of *equal length*

Two methods:

- Lowest common multiple
- Perpetuity method

[1] Lowest common multiple

- if Machine A has a life of 5 years and B has a life of 3 years, the lowest common multiple is **15 yrs**
- Machine A is replicated *3 times* and machine B is replicated *5 times* – the PV of the costs of both machines over *15yrs* are then compared

[2] Perpetuity method

- Both projects are replicated *forever* – the chains are then "equally lengthy" as both are infinite
- Is easier to use

$$
\begin{aligned}
\text{NPV}\infty \quad &= NPV + \frac{NPV}{(1+k)^t} + \frac{NPV}{(1+k)^{2t}} + \ldots\ldots \\
&= NPV\left[1 + \frac{1}{(1+k)^t} + \frac{1}{(1+k)^{2t}} + \ldots\ldots\right] \\
&= NPV\left[\frac{1}{1-\frac{1}{(1+k)^t}}\right] = NPV\,\frac{(1+k)^t}{(1+k)^t - 1}
\end{aligned}
$$

EXAMPLE: Using the prior example, where PV(costs of A) = \$664,949 where t = 4 and k = 0.13

$$
\begin{aligned}
\text{PV(A)}\infty \quad &= \$664\,949 \times \frac{(1.13)^4}{(1.13)^4 - 1} \\
&= \$1\,719\,631
\end{aligned}
$$

PV(Costs of B) = \$499 283; t = 2 and k = 0.13

$$
\begin{aligned}
\text{PV(B)}\infty \quad &= \$499\,283 \times \frac{(1.13)^2}{(1.13)^2 - 1} \\
&= \$2\,302\,400
\end{aligned}
$$

For an infinite chain of replacement, the costs of Machine B (\$.23m) are greater than costs fo A (\$1.7m) so should choose machine A

Equivalent Annual Value (EAV)

What amount, to be received for *n* years, is equivalent to receiving the NPV of a project whose life is *n years?*

$$
NPV = \frac{EAV}{k}\left(1 - \frac{1}{(1+k)^t}\right) \longrightarrow EAV = \frac{NPV}{\frac{1}{k}\left(1 - \frac{1}{(1+k)^t}\right)}
$$

1. *The PV of a **perpetual** stream of EAV's is given by:*

$$PV = \frac{EAV}{k} \longrightarrow EAV = \frac{NPV}{\frac{1}{k}\left(1 - \frac{1}{(1+k)^t}\right)} \longrightarrow PV = \left(\frac{1}{k}\right)\frac{NPV}{\frac{1}{k}\left(1 - \frac{1}{(1+k)^t}\right)}$$

2. *EAV and NPV:*

$$PV = NPV \frac{1}{\left(1 - \frac{1}{(1+k)^t}\right)}$$

$$= NPV \frac{(1+k)^t}{(1+k)^t - 1}$$

$$= NPV\infty$$

Note that the EAV and the perpetual chain of replacement methods are **equivalents** in regard to choosing a project

$$EAV = k \times NPV\infty$$

3. *EAV and EAC:*

- The principle of EAV is also applicable in the context of comparing the cost of investments of different lives (the *equivalent annual costs EAC)*
- EAC reflects the **annuity cost** that has the *same present value* as the NCFs of an investment over the investment period we are considering

$$EAC_i = kNPV_i\left[\frac{(1+k)^t}{(1+k)^t - 1}\right]$$

Where:
k is the opportunity cost of capital,
NPVi is the normal NPV of the investment i,
t is the life of the investment

EXAMPLE: Where you are considering replacing your old Nissan with a new Toyota (r = 3%);
- Annual maintenance costs for Nissan = $1500
- Annual maintenance costs of Toyota = $200
- Price of Toyota = 15yrs & remaining life of Nissan = 3 years

NPV (Toyota) = 18,000 + 200 x $\frac{1-(1.03)^{-15}}{0.03}$ = $20,387.59

EAC of Toyota = 0.03 x 20,387.59 x $\frac{(1.03)^{15}}{1.03^{15} - 1}$ = $1707.80

EAC of new car > $1500 SO the Nissan should be used for 3 more years (then buy Toyota)

Risk measures

Risk is the *uncertainty of future outcomes* or the *probability of an adverse outcome*

- Alternatively it is a chance of financial loss (the **variability of returns** associated with an asset)

Variance = $Var(R) = \frac{1}{T-1}[(R_1 - \bar{R})^2 + (R_2 - \bar{R})^2 + ... + (R_T - \bar{R})^2]$ **or** $Var(R) = \sigma^2 = \sum_{i=1}^{n} [R_i - E(R)]^2 p_i$

$where,\ p_i = \frac{1}{T-1}$

Standard deviation = $SD(R) = \sqrt{Var(R)}$

EXAMPLE: What is the standard deviation and variance for the following investment?

year 1: 19%
year 2: 20%
year 3: -30%
year 4: 26%

→ **sd = 0.26 & var = 0.0677**

EXAMPLE: Yearly returns for the following investment?

Year	**Price**	**Dividend**	**Returns**
1	31.64	0.1	-
2	32.09	0.1	0.0173831
3	34.99	0.17	0.0956684
4	23.02	0.17	-0.337239

(1) Average (do on calculator) – *sum/number years* = -0.0747

(2) HPR = (1 + 0.0174) (1 + 0.0957) (1 – 0.3372) – 1 =0.739

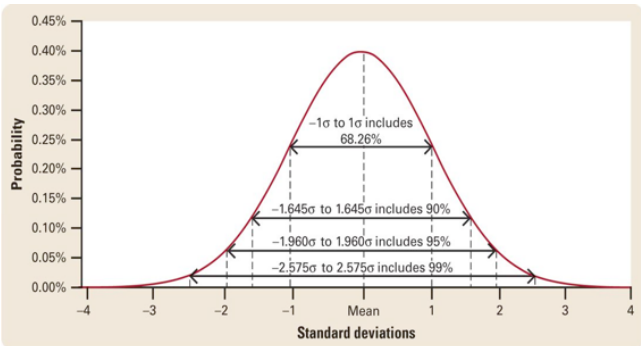
(3) SD = 0.2307 // Var = 0.05322

Normal distribution

Normal distribution: is completely described by its most likely outcome *μ (mean)* and *σ* (standard deviation)

Standard deviation = chance of getting around the average

- 68.26% chance that you get *1 SD around the mean*
- 95% chance you get *2 SD below/above the mean*



EXAMPLE: exam marks are normally distributed with *μ = 70%* and *σ = 10%*

(1) What percentage of student marks lie within *1 standard deviation from the mean?*
- **68.26%** of student marks lie between 60% and 80%.

(2) Between what bounds do we have 90% of the marks?
- 90% of students lie between:
  - 70% - (1.645*10%) AND 70% + (1.645*10%)
  - *Bounds are 53.55% and 86.45%*