Note that if f is a C^1 scalar function then:

$$\nabla f = \boldsymbol{i}\frac{\partial f}{\partial x} + \boldsymbol{j}\frac{\partial f}{\partial y} + \boldsymbol{k}\frac{\partial f}{\partial z}$$

Note that this implies that it is a vector field. If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is a C^1 vector field, the **divergence** of **F** is given by:

$$\begin{aligned}\text{div}\,\boldsymbol{F} &= \nabla \cdot \boldsymbol{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (F_1, F_2, F_3) \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.\end{aligned}$$

Note that div(**F**) is a scalar function.

Example 1

Find the divergence of **F.**

$$\begin{aligned}\boldsymbol{F}(x,y,z) &= x^2y\,\boldsymbol{i} + z\,\boldsymbol{j} + xyz\,\boldsymbol{k} \\ div(\boldsymbol{F}) &= \nabla . \boldsymbol{F} \\ &= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(xyz) \\ &= 2xy + 0 + xy \\ &= 3xy\end{aligned}$$

If **F** is the fluid velocity at position (x, y, z), then the divergence gives a measure of the net transport of fluid in/out of that point. If the divergence is greater than 0, then more fluid flows out than in, so the fluid is expanding. If the divergence is less than 0, then more fluid flows in than out, so the fluid is compressing. If the divergence is equal to 0, then the rate at which the fluid flows in equals the rate at which the fluid flows out. If the divergence is equal to 0, then **F** is called an **incompressible vector field**.

The **curl** of **F** is given by:

$$\begin{aligned}\text{curl}\,\boldsymbol{F} &= \nabla \times \boldsymbol{F} \\ &= \det\begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{bmatrix} \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\boldsymbol{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\boldsymbol{j} \\ &\quad + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k}.\end{aligned}$$

Note that the curl **F** is a vector field.

Example 2
Find the curl of **F**.

$$\boldsymbol{F}(x,y,z) = x^2y\,\boldsymbol{i} - 2xz\,\boldsymbol{j} + (x+y-z)\,\boldsymbol{k}$$

$$\begin{aligned} curl(\boldsymbol{F}) &= \nabla \times \boldsymbol{F} \\ &= (1+2x)\,\boldsymbol{i} - \boldsymbol{j} + (-x^2 - 2z)\,\boldsymbol{k} \end{aligned}$$

If **F** is the fluid velocity in a lake, and if one drops a twig into the lake, then the curl of **F** measures how quickly and in what orientation the twig rotates as it moves. If the twig does not rotate as it travels, then the curl of **F** is equal to **0** and is hence called **irrotational**. If the direction of the twig changes as it travels then the curl of **F** does not equal **0** and is called **rotational**. Note that if *f* is a scalar function the the divergence and curl of *f* are not defined; must be a vector field.

The **Laplacian Operator** is given by:

$$\begin{aligned} \nabla^2 &= \nabla \cdot \nabla \\ &= \left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right) \cdot \left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}\right) \\ &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \end{aligned}$$

Note that if *f* is a C^2 scalar function then:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

If **F** is a C^2 vector field where u, v and w are scalar functions of (x, y, z) then:

$$\nabla^2 \boldsymbol{F} = \nabla^2 u\,\boldsymbol{i} + \nabla^2 v\,\boldsymbol{j} + \nabla^2 w\,\boldsymbol{k}$$

$$\text{where } \nabla^2 u = u_{xx} + u_{yy} + u_{zz} \text{ etc.}$$

Note that this implies that the Laplacian Operator applied to **F** is a vector field. Some applications for the Laplacian Operator is the gravitational potential V of a mass *m* at (x, y, z) due to a mass M at (0, 0, 0), which satisfies Laplace's equation.