TRC4800 - Robotics Summary Notes

Spatial Transformation

Rotation Matrices have general form:

$$^A_B\mathbf{R} = \begin{bmatrix} ^A\mathbf{x}_B & ^A\mathbf{y}_B & ^A\mathbf{z}_B \end{bmatrix} = \begin{bmatrix} \mathbf{x}_B \cdot \mathbf{x}_A & \mathbf{y}_B \cdot \mathbf{x}_A & \mathbf{z}_B \cdot \mathbf{x}_A \\ \mathbf{x}_B \cdot \mathbf{y}_A & \mathbf{y}_B \cdot \mathbf{y}_A & \mathbf{z}_B \cdot \mathbf{y}_A \\ \mathbf{x}_B \cdot \mathbf{z}_A & \mathbf{y}_B \cdot \mathbf{z}_A & \mathbf{z}_B \cdot \mathbf{z}_A \end{bmatrix}$$

x-axis	y-axis	z-axis
$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$	$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$	$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

The magnitude of a rotation matrix is 1.

Transformation

Transformation is the combination of rotation and translation:

$$^A_BT = \begin{bmatrix} ^A_BR & ^A_BO \\ \tilde{0} & 1 \end{bmatrix}$$

Inverse of transformation matrix is used to transform in the opposite direction:

$$^B_AT = {^A_BT}^{-1} = \begin{bmatrix} ^A_BR^T & -^A_BR^T\,^A_BO \\ \tilde{0} & 1 \end{bmatrix}$$

Rotation/transformation matrices can have two meanings/applications:

- **Meaning 1 –** Map the coordiantes of a point in one frame into another frame. The point can be measured in either frame, but position vector will differ for each case. The transformation matrix represents the transformation between the two frames.
  - Point remains constant, measure frame changes

$$^1p = {^1_2T} \times {^2p}$$

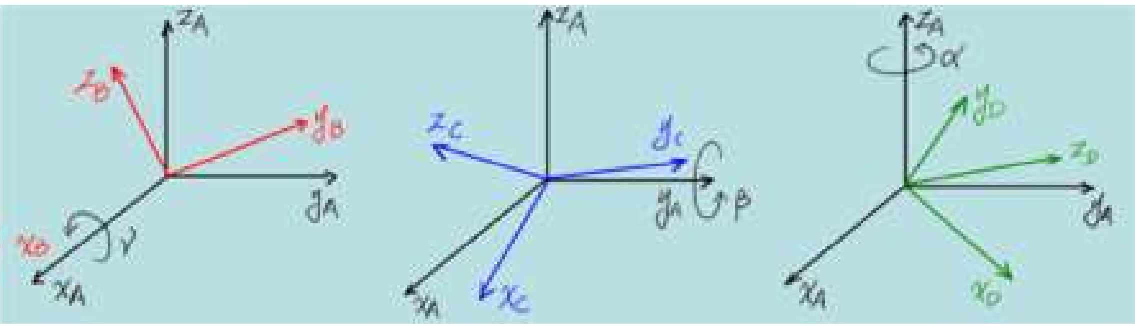
- **Meaning 2 –** A point is only measured in a base frame but is transformed to a new position defined by the transformation matrix
  - Base frame remains constant, point position changes

$$^1p_2 = T \times {^1p_1}$$

When multiple transformations are applied, the base/measurement frame can remain constant (meaning 2) or change with each step (meaning 1).

Fixed Angles

Any arbitrary orientation of an object can be achieved by a sequence of rotations around a fixed base frame A:

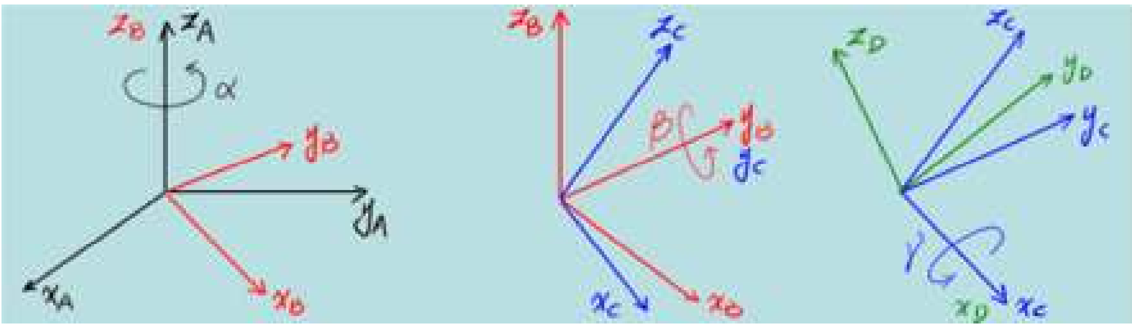


Transformation matrices are multipled in inverse order to which they are applied:

$$^{A}_{D}\mathbf{R}_{XYZ}(\gamma,\beta,\alpha) = ^{A}_{D}\mathbf{R}_{z}(\alpha)^{A}_{C}\mathbf{R}_{y}(\beta)^{A}_{B}\mathbf{R}_{x}(\gamma)$$

Euler Angles

At each transformation step, the transformed frame becomes the base frame for the next transformation.



In this case, transformation matrices are multiplied in the same order as they are applied:

$$^{A}_{D}\mathbf{R}_{XYZ}(\alpha,\beta,\gamma) = ^{A}_{B}\mathbf{R}_{z}(\alpha)^{B}_{C}\mathbf{R}_{y}(\beta)^{C}_{D}\mathbf{R}_{x}(\gamma)$$

Denavit-Hartenberg Notation

DH-convention is used to define rotation and translation parameters of robotic manipulator links.

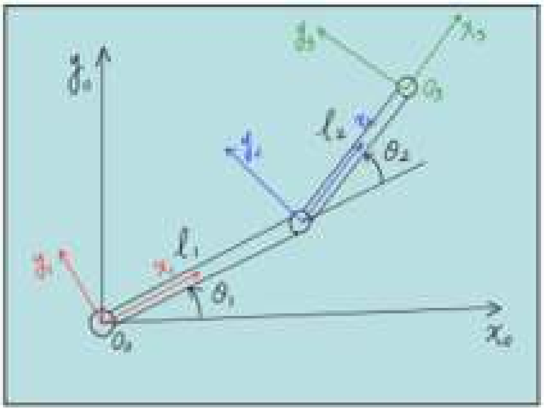
- a_{i-1} is the link distance between two axes z_{i-1} and z_i along x_{i-1}.
- α_{i-1} is the twist angle between two axes z_{i-1} and z_i along x_{i-1}.
- d_i is the offset between two axes x_{i-1} and x_i along z_i.
- θ_i is the joint angle between two axes x_{i-1} and x_i along z_i.

These are then used to generate the transformation matrix between the two frames:

$$^{i-1}_{i}T = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & a_{i-1} \\ \sin\theta_i\cos\alpha_{i-1} & \cos\theta_i\cos\alpha_{i-1} & -\sin\alpha_{i-1} & -\sin\alpha_{i-1}d_i \\ \sin\theta_i\sin\alpha_{i-1} & \cos\theta_i\sin\alpha_{i-1} & \cos\alpha_{i-1} & \cos\alpha_{i-1}d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

Assume point masses at joint location, zero force on end effector.



$$ {}^1\mathbf{p}_{c1} = \begin{bmatrix} l_1 & 0 & 0 \end{bmatrix}^T, \quad {}^2\mathbf{p}_{c2} = \begin{bmatrix} l_2 & 0 & 0 \end{bmatrix}^T $$

$$ {}^0\ddot{\mathbf{O}}_0 = \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix} \qquad {}^1_0\boldsymbol{\omega}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \qquad {}^1_0\dot{\boldsymbol{\omega}}_1 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix} $$

$$ \begin{aligned} {}^1_0\ddot{\mathbf{O}}_1 &= {}^1_0\mathbf{R}\,({}^0_0\ddot{\mathbf{O}}_0 + {}^0_0\dot{\boldsymbol{\omega}}_0 \times {}^0_0\mathbf{O}_1 + {}^0_0\boldsymbol{\omega}_0 \times ({}^0_0\boldsymbol{\omega}_0 \times {}^0_0\mathbf{O}_1)) \\ &= \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix} = \begin{bmatrix} g s_1 \\ g c_1 \\ 0 \end{bmatrix} \end{aligned} $$

Mass centre 1 acceleration:

$$ \begin{aligned} {}^1_0\ddot{\mathbf{p}}_{c1} &= {}^1_0\ddot{\mathbf{O}}_1 + {}^1_0\dot{\boldsymbol{\omega}}_1 \times {}^1_1\mathbf{p}_{c1} + {}^1_0\boldsymbol{\omega}_1 \times ({}^1_0\boldsymbol{\omega}_1 \times {}^1_1\mathbf{p}_{c1}) \\ &= \begin{bmatrix} -l_1\dot{\theta}_1^2 + g s_1 \\ l_1\ddot{\theta}_1 + g c_1 \\ 0 \end{bmatrix} \end{aligned} $$

Used to find force and torque at mass centre (torque will be 0 since point mass):

$$ {}^1\mathbf{f}_{c1} = m_1\,{}^1\ddot{\mathbf{p}}_{c1} = \begin{bmatrix} -m_1 l_1\dot{\theta}_1^2 + m_1 g s_1 \\ m_1 l_1\ddot{\theta}_1 + m_1 g c_1 \\ 0 \end{bmatrix} \qquad {}^1\mathbf{n}_{c1} = {}^1\mathbf{I}_{c1}\,{}^1\dot{\boldsymbol{\omega}}_1 + {}^1\boldsymbol{\omega}_1 \times {}^1\mathbf{I}_{c1}\,{}^1\boldsymbol{\omega}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$

Mass centre 2:

$$ {}^2_0\boldsymbol{\omega}_2 = {}^2_1\mathbf{R}\,{}^1_0\boldsymbol{\omega}_1 + \dot{\theta}_2\,{}^2\mathbf{z}_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} $$

$$ {}^2_0\dot{\boldsymbol{\omega}}_2 = {}^2_1\mathbf{R}\,{}^1_0\dot{\boldsymbol{\omega}}_1 + ({}^2_1\mathbf{R}\,{}^1_0\boldsymbol{\omega}_1) \times (\dot{\theta}_2\,{}^2\mathbf{z}_2) + \ddot{\theta}_2\,{}^2\mathbf{z}_2 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 \end{bmatrix} $$

$$ {}^2_0\ddot{\mathbf{O}}_2 = {}^2_1\mathbf{R}\,({}^1_0\ddot{\mathbf{O}}_1 + {}^1_0\dot{\boldsymbol{\omega}}_1 \times {}^1_0\mathbf{O}_2 + {}^1_0\boldsymbol{\omega}_1 \times ({}^1_0\boldsymbol{\omega}_1 \times {}^1_0\mathbf{O}_2)) = \begin{bmatrix} l_1\ddot{\theta}_1 s_2 - l_1\dot{\theta}_1^2 c_2 + g s_{12} \\ l_1\ddot{\theta}_1 c_2 + l_1\dot{\theta}_1^2 s_2 + g c_{12} \\ 0 \end{bmatrix} $$

Mass centre acceleration and forces (torque will again be zero):

$$ \begin{aligned} {}^2_0\ddot{\mathbf{p}}_{c2} &= {}^2_0\ddot{\mathbf{O}}_2 + {}^2_0\dot{\boldsymbol{\omega}}_2 \times {}^2_1\mathbf{p}_{c2} + {}^2_0\boldsymbol{\omega}_2 \times ({}^2_0\boldsymbol{\omega}_2 \times {}^2_1\mathbf{p}_{c2}) \\ &= \begin{bmatrix} -l_2(\dot{\theta}_1 + \dot{\theta}_2)^2 + l_1\ddot{\theta}_1 s_2 - l_1\dot{\theta}_1^2 c_2 + g s_{12} \\ l_2(\ddot{\theta}_1 + \ddot{\theta}_2) + l_1\ddot{\theta}_1 c_2 + l_1\dot{\theta}_1^2 s_2 + g c_{12} \\ 0 \end{bmatrix} \end{aligned} $$

$$ {}^2\mathbf{f}_{c2} = m_2\,{}^2\ddot{\mathbf{p}}_{c2} = \begin{bmatrix} -m_2 l_2(\dot{\theta}_1 + \dot{\theta}_2)^2 + m_2 l_1\ddot{\theta}_1 s_2 - m_2 l_1\dot{\theta}_1^2 c_2 + m_2 g s_{12} \\ m_2 l_2(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1\ddot{\theta}_1 c_2 + m_2 l_1\dot{\theta}_1^2 s_2 + m_2 g c_{12} \\ 0 \end{bmatrix} $$