

ENG2092 - Advanced Maths B Summary Notes

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Complex Analysis

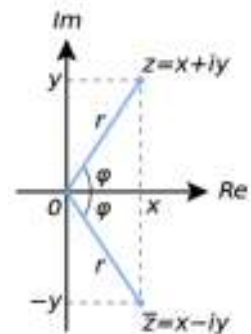
Concepts

Conjugate

Conjugate of a complex number is achieved by changing the sign of the imaginary component.

- Geometrically represented as a reflection about the x-axis

$$\bar{z} = \overline{x + iy} := x - iy$$



Modulus

The magnitude or length of the vector.

$$|z| = |x + iy| := \sqrt{x^2 + y^2}$$

$$|z|^2 = z\bar{z}.$$

Division

$$\frac{z}{w} := \frac{z\bar{w}}{|w|^2}$$

$$(i) \quad |w + z| \leq |w| + |z|,$$

$$(ii) \quad |z - w| \geq ||z| - |w||,$$

$$(iii) \quad |\operatorname{Re}(z)| \leq |z|, \quad |\operatorname{Im}(z)| \leq |z|$$

$$(iv) \quad |\bar{z}| = |z|, \text{ and}$$

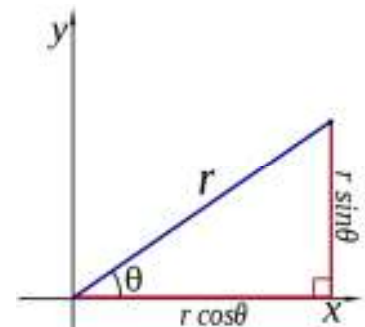
$$(v) \quad |wz| = |z||w|$$

Polar Form

On complex plane, polar coordinates can be represented by:

$$x = r \cos(\theta) \quad y = r \sin(\theta),$$

$$r = \sqrt{x^2 + y^2} \quad \theta = \arctan\left(\frac{y}{x}\right)$$



So complex number written as:

$$z = r(\cos(\theta) + i \sin(\theta)) = re^{i\theta} \quad (\text{Euler's Formula})$$

$$|e^{i\theta}| = 1, \quad e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi}$$

Converting to Polar Form

$$\theta = \arg(z) = \arctan\left(\frac{y}{x}\right)$$

$$r = |z| = \sqrt{x^2 + y^2}$$

Operations in polar form

$$z_1 z_2 = (r_1 e^{i\theta_1} r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1 + \theta_2)} \quad \frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

Delta Function

Function which models an impulse, has properties:

- $\delta(t - a) = 0$ for all $t \neq a$
- Has integral equal to 1 over any range which includes $t = a$

For example a function of constant velocity given a short sharp acceleration of Δv :

$$v(t) = v_0 + \int_0^t a(\tau) d\tau = v_0 + \int_0^t (\Delta v) \delta(\tau - 1) d\tau = v_0 + (\Delta v) \int_0^t \delta(\tau - 1) d\tau$$

- $v(t) = v_0$ for $t < 1$
- $v(t) = v_0 + \Delta v$ for $t > 1$

Is able to 'pick out' values of the integrand of an integral:

$$\int_0^\infty g(t) \delta(t - a) dt = g(a) \text{ for any function } g(t)$$

$$\mathcal{L}(\delta(t - a)) = e^{-as} \text{ for any } a > 0$$

Convolution

A new operation between two functions f and g is called **convolution** and defined as:

$$(f * g)(t) = \int_0^t f(\tau) g(t - \tau) d\tau$$

And can be shown that:

$$\mathcal{L}(f * g) = \mathcal{L}(f) \mathcal{L}(g) = F(s) G(s)$$

Limits of Transformed Functions

The Laplace transform of a function can be used to identify properties of the original function before inversion.

If the Laplace transforms of both $f(t)$ and $f'(t)$ exist, and $\lim_{s \rightarrow \infty} [sF(s)]$ exists, then

$$\lim_{t \rightarrow 0^+} f(t) = \lim_{s \rightarrow \infty} [sF(s)].$$

If the Laplace transforms of both $f(t)$ and $f'(t)$ exist, and $\lim_{t \rightarrow \infty} f(t)$ exists, then

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} [sF(s)].$$

In some applications there may have been an impulse or jump at $t=0$, so $f(0)$ is not always the same as the limit $f(t \rightarrow 0)$.

Transfer Functions

Solving the second order DE:

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = v_i(t)$$

Gives:

$$(Ls^2 + Rs + \frac{1}{C})Q(s) = V_i(s)$$

And so:

$$Q(s) = H_q(s) V_i(s) \quad H_q(s) = \frac{C}{CLs^2 + RCs + 1}$$

The $H(s)$ function is known as the **transfer function** (factor between $Q(s)$ and $V(s)$).

- Determines the **output strength** directly from the transforms of the input forcing function.

And then using convolution:

$$q(t) = (v_i * h_q)(t) = \int_0^t v_i(\tau) h_q(t - \tau) d\tau$$

Response to Impulse Forcing

Given forcing function $f(t) = f_0 \delta(t - a)$, $y(t) = f_0 h_y(t - a)$ for $t > a$

Response to Periodic Forcing

Periodic behaviour is more effectively analysed using **Fourier series and Fourier transforms**.

Example of periodic function:

$$f(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]$$

Must be defined for all values of t and has property: $f(t + T_0) = f(t)$ for all t

- With **period** T_0
- Frequency $\frac{1}{T_0}$ (Hz)
- Angular frequency $\omega_0 = \frac{2\pi}{T_0}$ (rad/s)

Fourier Series

Above expression for trigonometric series has constants determined by the **Euler Formulae**:

$$a_0 = \frac{1}{T_0} \int_0^{T_0} f(t) dt, \quad a_n = \frac{2}{T_0} \int_0^{T_0} f(t) \cos(n\omega_0 t) dt \quad \text{and} \quad b_n = \frac{2}{T_0} \int_0^{T_0} f(t) \sin(n\omega_0 t) dt$$

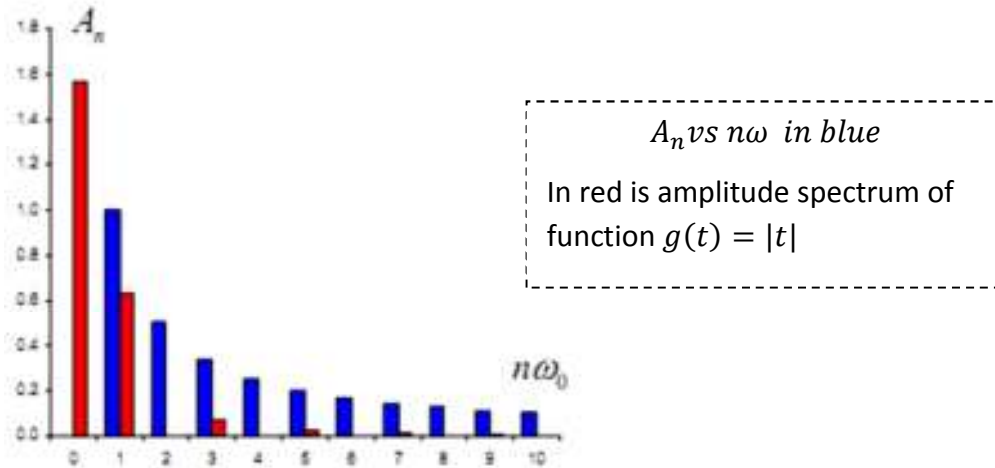
These are called the **Fourier coefficients**.

- The series shows the relative contribution of the **modes** or **harmonics**, each of which has angular frequency $\omega_n = n\omega_0$

Amplitude Spectrum

If the complex-valued coefficients are expressed in their polar form, then:

$$c_n = A_n e^{i\phi_n} \text{ where } A_n = |c_n| \text{ and } \phi_n = \arg(c_n) \quad f(t) = \sum_{n=-\infty}^{\infty} A_n e^{i(n\omega_0 t + \phi_n)}$$



Complex exponentials and Real functions

For real-valued function $f(t) = a \cos(\omega t) + b \sin(\omega t)$, equivalent complex-valued function:

$$f(t) = A e^{i(\omega t + \phi)} \quad - \text{ then analysis will be easier to conduct on this form.}$$

At the end, take real part of this to get: $\text{Re}\{f(t)\} = \text{Re}\{A e^{i(\omega t + \phi)}\} = A \cos(\omega t + \phi)$

Damped Linear Systems with Periodic Forcing

Damped vibrating mechanical system under an applied forcing is modelled by:

$$m \frac{d^2 y}{dt^2} + c \frac{dy}{dt} + k y = f(t) \quad \text{with forcing function of form } f(t) = A e^{i(\omega t + \phi)}$$

This will lead to a complex-valued solution $y(t)$ with similar polar form. Determining general first and second derivatives and substituting into equation gives:

$$[m(i\omega)^2 + c(i\omega) + k] B e^{i(\omega t + \theta)} = A e^{i(\omega t + \phi)}$$

Re-arranging leads to the transfer function:

$$B e^{i\theta} = \frac{A e^{i\phi}}{m(i\omega)^2 + c(i\omega) + k} = H_y(i\omega) A e^{i\phi}$$

The solution is the **real** part of:

$$y(t) = \sum_{n=-\infty}^{\infty} B_n e^{i(n\omega_0 t + \theta_n)} = \sum_{n=-\infty}^{\infty} H(in\omega_0) A_n e^{i(n\omega_0 t + \phi_n)}$$