ENG2092 - Advanced Maths B Summary Notes

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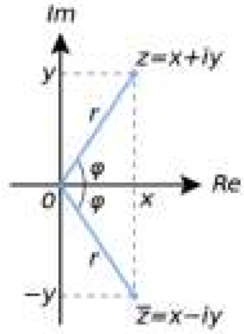
Complex Analysis

Concepts

Conjugate



Conjugate of a complex number is achieved by changing the sign of the imaginary component.

- Geometrically represented as a reflection about the x-axis

$$\bar{z} = \overline{x + iy} := x - iy$$

Modulus

The magnitude or length of the vector.

$$|z| = |x + iy| := \sqrt{x^2 + y^2}$$

$$|z|^2 = z\bar{z}.$$

(i) $|w + z| \le |w| + |z|$,

(ii) $|z - w| \ge ||z| - |w||$,

(iii) $|\mathrm{Re}(z)| \le |z|$, $|\mathrm{Im}(z)| \le |z|$

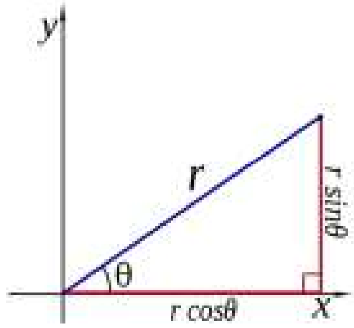
(iv) $|\bar{z}| = |z|$, and

(v) $|wz| = |z||w|$

Division

$$\frac{z}{w} := \frac{z\bar{w}}{|w|^2}$$

Polar Form



On complex plane, polar coordinates can be represented by:

$$x = r\cos(\theta) \quad y = r\sin(\theta),$$

$$r = \sqrt{x^2 + y^2} \quad \theta = \arctan\left(\frac{y}{x}\right)$$

So complex number written as:

$$z = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}$$ (Euler's Formula)

$$|e^{i\theta}| = 1, \quad e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi}$$

Converting to Polar Form

$$\theta = \arg(z) = \arctan\left(\frac{y}{x}\right) \qquad r = |z| = \sqrt{x^2 + y^2}$$

Operations in polar form

$$z_1 z_2 = (r_1 e^{i\theta_1} r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)} \qquad \frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

Delta Function

Function which models an impulse, has properties:

- $\delta(t-a) = 0$ for all $t \neq a$
- Has integral equal to 1 over any range which includes t = a

For example a function of constant velocity given a short sharp acceleration of Δv:

$$v(t) = v_0 + \int_0^t a(\tau)\,d\tau = v_0 + \int_0^t (\Delta v)\,\delta(\tau-1)\,d\tau = v_0 + (\Delta v)\int_0^t \delta(\tau-1)\,d\tau$$

- $v(t) = v_0 \qquad for\ t < 1$
- $v(t) = v_0 + \Delta v \quad for\ t > 1$

Is able to 'pick out' values of the integrand of an integral:

$$\int_0^\infty g(t)\,\delta(t-a)\,dt = g(a) \text{ for any function } g(t)$$

$$\boxed{\mathcal{L}(\delta(t-a)) = e^{-sa} \quad \text{for any } a > 0}$$

Convolution

A new operation between two functions f and g is called **convolution** and defined as:

$$(f * g)(t) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

And can be shown that:

$$\mathcal{L}(f * g) = \mathcal{L}(f)\,\mathcal{L}(g) = F(s)G(s)$$

Limits of Transformed Functions

The Laplace transform of a function can be used to identify properties of the original function before inversion.

If the Laplace transforms of both $f(t)$ and $f'(t)$ exist, and $\lim_{s\to\infty}[sF(s)]$ exists, then $\lim_{t\to 0+} f(t) = \lim_{s\to\infty}[sF(s)]$.

If the Laplace transforms of both $f(t)$ and $f'(t)$ exist, and $\lim_{t\to\infty} f(t)$ exists, then $\lim_{t\to\infty} f(t) = \lim_{s\to 0}[sF(s)]$.

In some applications there may have been an impulse or jump at t=0, so f(0) is not always the same as the limit f(t->0).

Transfer Functions

Solving the second order DE:

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}=v_i(t)$$

Gives:

$$(Ls^2+Rs+\frac{1}{C})Q(s)=V_i(s)$$

And so:

$$Q(s)=H_q(s)V_i(s) \qquad H_q(s)=\frac{C}{CLs^2+RCs+1}$$

The H(s) function is known as the **transfer function** (factor between Q(s) and and V(s).

- Determines the **output strength** directly from the transforms of the input forcing function.

And then using convolution:

$$q(t)=(v_i*h_q)(t)=\int_0^t v_i(\tau)h_q(t-\tau)\,d\tau$$

Response to Impulse Forcing

Given forcing function $f(t)=f_0\,\delta(t-a)$, $y(t)=f_0\,h_y(t-a)$ for $t>a$

Response to Periodic Forcing

Periodic behaviour is more effectively analysed using **Fourier series and Fourier transforms.**

Example of periodic function:

$$f(t)=a_0+\sum_{n=1}^{\infty}[a_n\cos(n\,\omega_0\,t)+b_n\sin(n\,\omega_0\,t)]$$

Must be defined for all values of t and has property: $f(t+T_0)=f(t)$ for all t

- With **period** T_0
- Frequency $\frac{1}{T_0}$ (Hz)
- Angular frequency $\omega_0=\frac{2\pi}{T_0}$ (rad/s)

Fourier Series

Above expression for trigonometric series has constants determined by the **Euler Formulae:**

$$a_0=\frac{1}{T_0}\int_0^{T_0}f(t)\,dt, \quad a_n=\frac{2}{T_0}\int_0^{T_0}f(t)\cos(n\,\omega_0\,t)\,dt \quad \text{and} \quad b_n=\frac{2}{T_0}\int_0^{T_0}f(t)\sin(n\,\omega_0\,t)\,dt$$

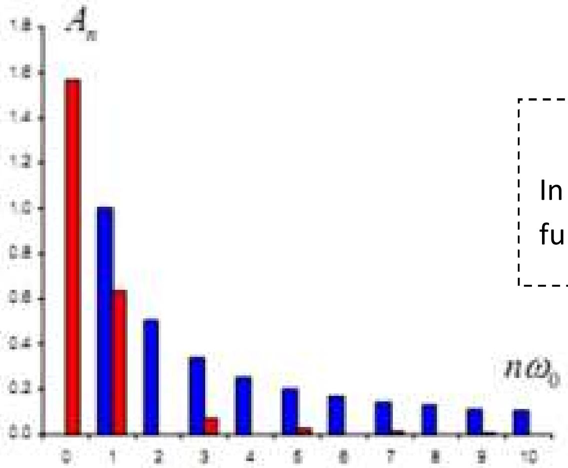
These are called the **Fourier coefficients.**

- The series shows the relative contribution of the **modes** or **harmonics**, each of which has angular frequency $\omega_n=n\omega_0$

Amplitude Spectrum

If the complex-valued coefficients are expressed in their polar form, then:

$c_n = A_n e^{i\phi_n}$ where $A_n = |c_n|$ and $\phi_n = \arg(c_n)$ $\qquad f(t) = \sum_{n=-\infty}^{\infty} A_n e^{i(n\omega_0 t + \phi_n)}$



A_n vs $n\omega$ in blue

In red is amplitude spectrum of function $g(t) = |t|$

Complex exponentials and Real functions

For real-valued function $f(t) = a\cos(\omega t) + b\sin(\omega t)$, equivalent complex-valued function:

$f(t) = A e^{i(\omega t + \phi)}$ - then analysis will be easier to conduct on this form.

At the end, take real part of this to get: $\mathrm{Re}\{f(t)\} = \mathrm{Re}\{A e^{i(\omega t + \phi)}\} = A\cos(\omega t + \phi)$

Damped Linear Systems with Periodic Forcing

Damped vibrating mechanical system under an applied forcing is modelled by:

$$m\frac{d^2 y}{dt^2} + c\frac{dy}{dt} + k\,y = f(t)$$ with forcing function of form $f(t) = A e^{i(\omega t + \phi)}$

This will lead to a complex-valued solution y(t) with similar polar form. Determining general first and second derivatives and substituting into equation gives:

$$[m(i\omega)^2 + c(i\omega) + k]\,B e^{i(\omega t + \theta)} = A e^{i(\omega t + \phi)}$$

Re-arranging leads to the transfer function:

$$B e^{i\theta} = \frac{A e^{i\phi}}{m(i\omega)^2 + c(i\omega) + k} = H_y(i\omega) A e^{i\phi}$$

The solution is the **real** part of:

$$y(t) = \sum_{n=-\infty}^{\infty} B_n e^{i(n\omega_0 t + \theta_n)} = \sum_{n=-\infty}^{\infty} H(in\omega_0) A_n e^{i(n\omega_0 t + \phi_n)}$$