MECH3200 Revision Document
Advanced Dynamics & Vibrations

Contents

Section A: Discrete Systems

1 Introduction

Discrete systems refer to systems which can realistically be modelled with discrete mass, damping and stiffness elements. We will be dealing with multiple degree of freedom systems (MDoFS). Two degree of freedom systems (2DoFS) will be the most common, as they typify the general case but are easier to solve by hand.

2 SDoF reminders

2.1 Adding spring stiffnesses

Springs in series: These are springs which are joined end to end, so their individual deflection adds to the total deflection. Since stiffness is proportional to 1/deflection, we have:

$$\frac{1}{k_{\text{tot}}} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n}$$

Springs in parallel: Adding more springs to a structure (eg. springs along a rigid body) means that we're adding stiffness to the supports, so we directly add the springs in series.

$$k_{\text{tot}} = k_1 + k_2 + \cdots + k_n$$

3 Solving a MDoF system

The solution process in a MDoF system typically involves:

1. Deriving the equations of motion for the system and putting them in matrix form.
2. Finding the natural frequencies and mode shapes of the system. (Skip this for direct analysis.)
3. Investigating damping or forcing effects, either by modal analysis or direct analysis.

3.1 Equations of motion through Lagrange's equation (A1)

Lagrange's equation is used to find the equations of motion (EoMs) of the system. In a MDoFS, it is usually easier to derive the EoMs based on energy terms, rather than considering free body diagrams. The general form of Lagrange's equation is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = \frac{\partial W}{\partial q_i}$$

where q_i is the generalised coordinate, $L = T - V$ is the Lagrangian, T is kinetic energy of the system, V is potential energy of the system, D is the power dissipated by viscous damping, and W is the work of applied loads (typically 0 in the systems we encounter here).

3.1.1 Typical expressions for Lagrange equation energy terms

Kinetic energy of rigid bodies	$T = \sum \frac{1}{2} m_i \dot{x}_i^2 + \sum \frac{1}{2} I_i \dot{\theta}_i^2$
Potential energy of linear springs	$V = \frac{1}{2} k_{ij} (x_i - x_j)^2$
Power dissipated by viscous damping	$D = \frac{1}{2} c_{ij} (\dot{x}_i - \dot{x}_j)^2$

3.2 Natural frequencies and mode shapes (A2)

There are as many EoMs as degrees of freedom in the system. This enables us to represent the system of equations as a single equation in matrix form. This system is then solved to find the natural frequencies and mode shapes of the system (unforced, undamped solution), which typically serve as a basis for other solutions (eg. the addition of damping or forcing terms).

3.2.1 Matrix formulation

The general form of matrix representation of a 2DoFS is:

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix}$$

In general:

$$M\ddot{\mathbf{q}} + C\dot{\mathbf{q}} + K\mathbf{q} = \mathbf{F}$$

where M, C and K are the mass, damping and stiffness matrices repsectively, $\mathbf{q}$ is the coordinate vector and $\mathbf{F}$ is the force vector.

3.2.2 Free, undamped vibration solution

The equation for free, undamped vibration is:

$$M\ddot{\mathbf{q}} + K\mathbf{q} = 0$$

The unforced and undamped motion is governed by the equation:

$$\mathbf{q} = \mathbf{Q} \cos(\omega t + \varphi)$$

where $\mathbf{Q}$ is the amplitude of oscillation, ω is the frequency of vibration (for unforced oscillation, this is the natural frequency), and φ is the phase angle.

Hence, $\ddot{\mathbf{q}}$ is given by the following expression:

$$\ddot{\mathbf{q}} = -\omega^2 \mathbf{Q} \cos(\omega t + \varphi) = -\omega^2 \mathbf{q}$$

This allows us to rearrange the equations of motion:

$$\begin{aligned} -\omega^2 M\mathbf{q} + K\mathbf{q} &= 0 \\ [K - \omega^2 M]\mathbf{q} &= 0 \\ [K - \omega^2 M] \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \cos(\omega t + \varphi) &= 0 \\ [K - \omega^2 M] \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} &= 0 \quad \text{because the equation must always be satisfied} \end{aligned}$$

Akin to an eigenvalue problem, there are two solutions to this equation - the trivial solution of 0 vibration amplitude, or the non-trivial solution representing vibration, which requires:

$$|K - \omega^2 M| = 0$$