MEC4418 - Modelling and Control Summary Notes

Contents

Introduction

Transfer Function

The transfer function of a linear, time-invariant, differential equation system is defined as the ratio of the Laplace transform of the output (response) to the Laplace transform of the input (driving function) under the assumption that all initial conditions are zero.

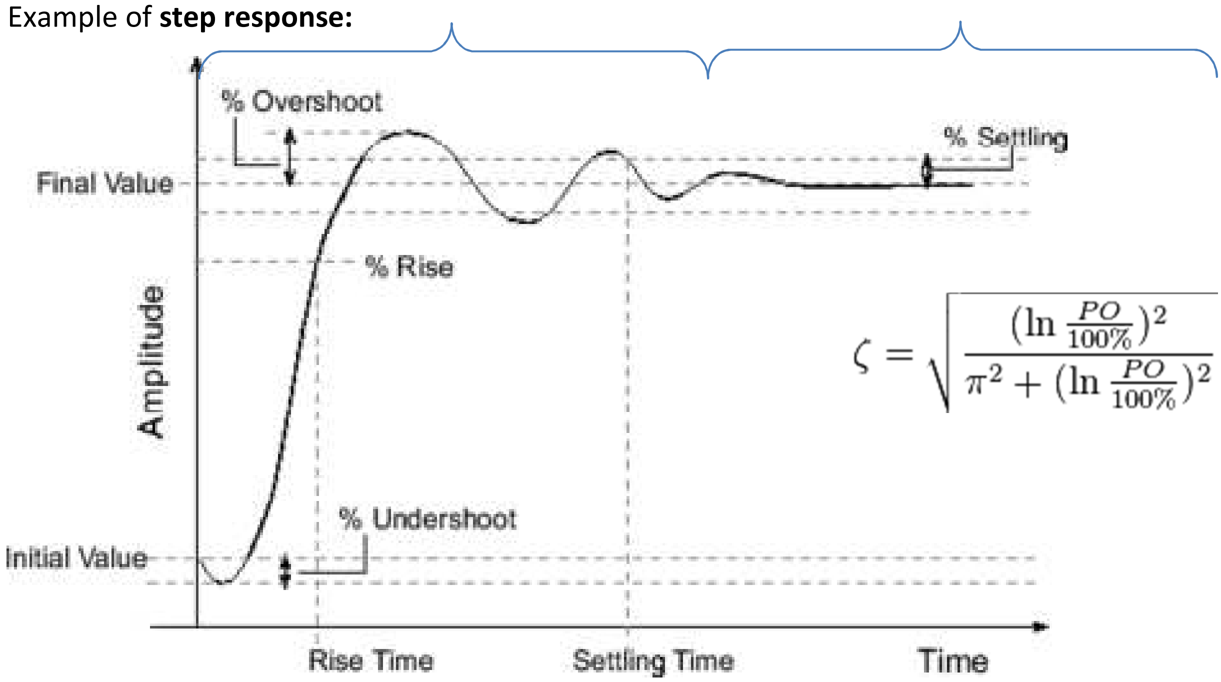
- System dynamics are represented by algebraic equations in 's'

Response

The output (time domain response) of a system consists of two parts:

Transient	Steady-State
transition from intitial state to the final state (involves rise time, overshoot, settling time)	Behaviour of system as time approaches infinity

Example of **step response:**



$$\zeta = \sqrt{\frac{(\ln \frac{PO}{100\%})^2}{\pi^2 + (\ln \frac{PO}{100\%})^2}}$$

Transient Response Specifications

Peak time	Peak value (maximum overshoot)	Settling time (within 2%)
$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$	$M_p = e^{-\left(\frac{\xi\pi}{\sqrt{1-\xi^2}}\right)} \ (\times 100\%)$	$t_s = \frac{4}{\xi\omega_n}$

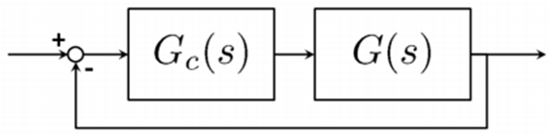
Stability

Stable – output eventually returns to equilibrium state when system is subjeced to intial condition

Critically stable – oscillations of output continue forever

Unstable – Output diverges without bound when system subjected to initial condition

In the following compensator discussions, a unit-feedback structure is assumed.



Lead Compensation

Useful to modify **transient-response** characteristics.

- Desired closed-loop pole locations determined using given specifications

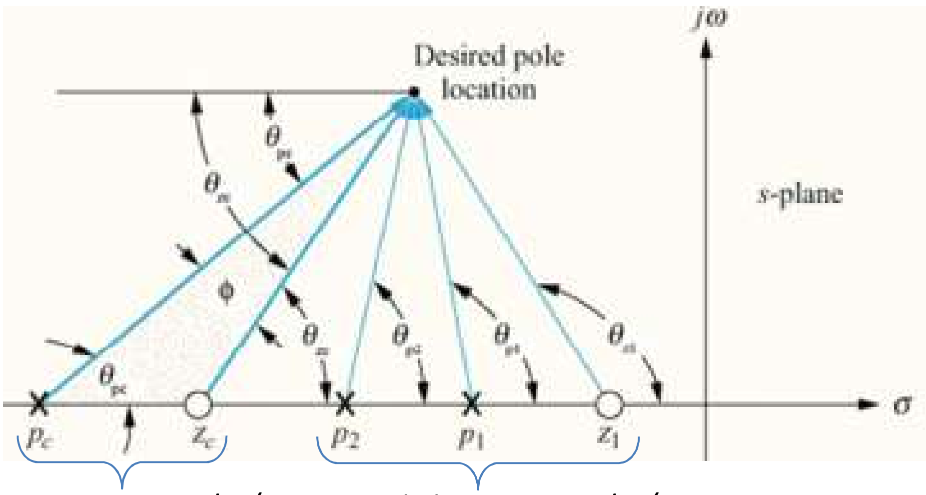
General format (pole larger than zero):

$$G_{ld}(s) = K_{ld}\frac{s + 1/T}{s + \frac{1}{\alpha T}},\ 0 < \alpha < 1$$

The **magnitude** and **angle** conditions are used to determine the compensator zero and pole.

- The angle contribution or **angle deficiency** of the lead compensator must be such that the angle condition is satisfied at the desired closed-loop pole

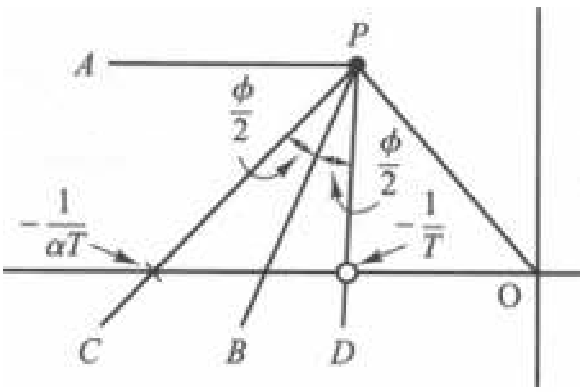
$$\emptyset + \sum \theta_z - \sum \theta_p = \pm 180(2k+1) \qquad \emptyset = \theta_{zc} - \theta_{pc}$$



Required compensator poles/zeros *Existing system poles/zeros*

As can be seen, there are infinitely many choices of p_c and z_c that have the same required angle difference between them. Angle bisection method can be used to find ideal locations:

- Line PB bisects angle APO
- Angle deficiency Ø is also bisected by PB to produce lines PC and PD
- Intersections of these lines with real axis produce the desired pole/zero

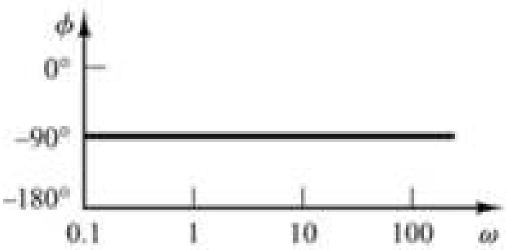


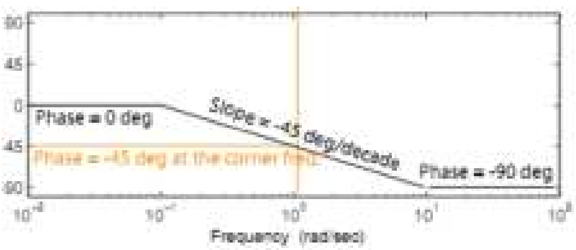
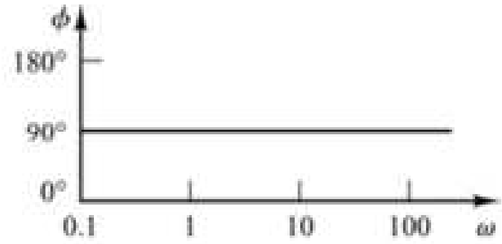
Magnitude Condition is then used to find the required gain 'K'.

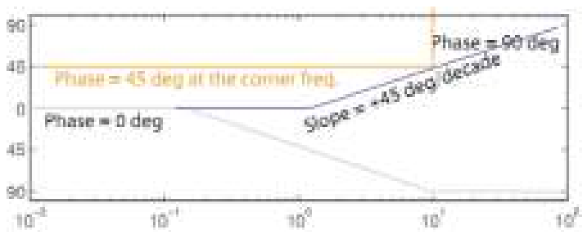
$$|K_{ld}G_{ld}(s)G(s)|_{s=CLP} = 1$$

Phase Plot

Gain has no effect on phase plot

<table>
<tr><th>Poles</th><th>Zeros</th></tr>
<tr>
<td>Pole at origin causes -90° vertical shift of starting angle


First-order pole causes a -45°/dec slope beginning at $\frac{\omega_p}{10}$ and lasting for 2 decades until $10\omega_p$

</td>
<td>Pole at origin causes +90° vertical shift of starting angle


First-order zero causes a +45°/dec slope beginning at $\frac{\omega_z}{10}$ and lasting for 2 decades until $10\omega_z$

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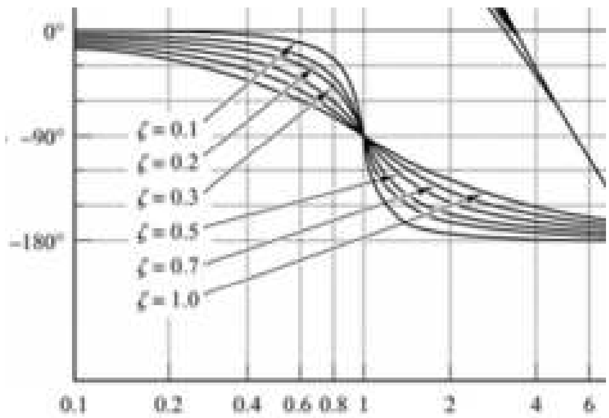
Second order pole/zero causes -/+90°/dec slope beginning at $\frac{\omega_n}{10}$ and lasting for 2 decades

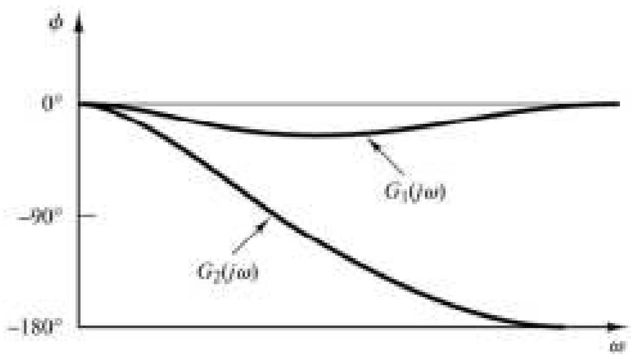
- Non-asymptotic behaviour around ω_n is a function of ξ
- Calculate phase for frequencies **around** ω_n to determine nature (use symmetry)



Minimum-Phase Systems

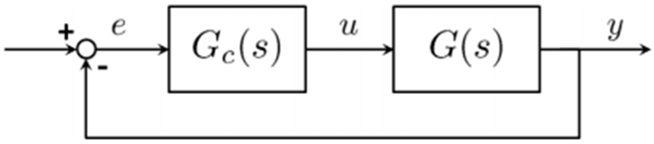
A **minimum-phase** transfer function does not have poles or zeros in the RHP.

- Non-minimum-phase systems have **faulty transient behaviours** and are difficult to deal with
- Non-minimum-phase systems have large **variations in phase angle**
- Two systems shown have same magnitude characteristics, but G_2 is a non-minimum-phase system.



PID Controllers

PID controller is a simple and quick solution to control a system.



The PID controller $G_c(s)$ operates on the **error signal** 'e' to produce an input 'u' to the plant $G(s)$.

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau)d\tau + K_d \frac{de}{dt}(t) = K_p\left\{e(t) + \frac{1}{T_i}\int_0^t e(\tau)d\tau + T_d \frac{de}{dt}(t)\right\}$$

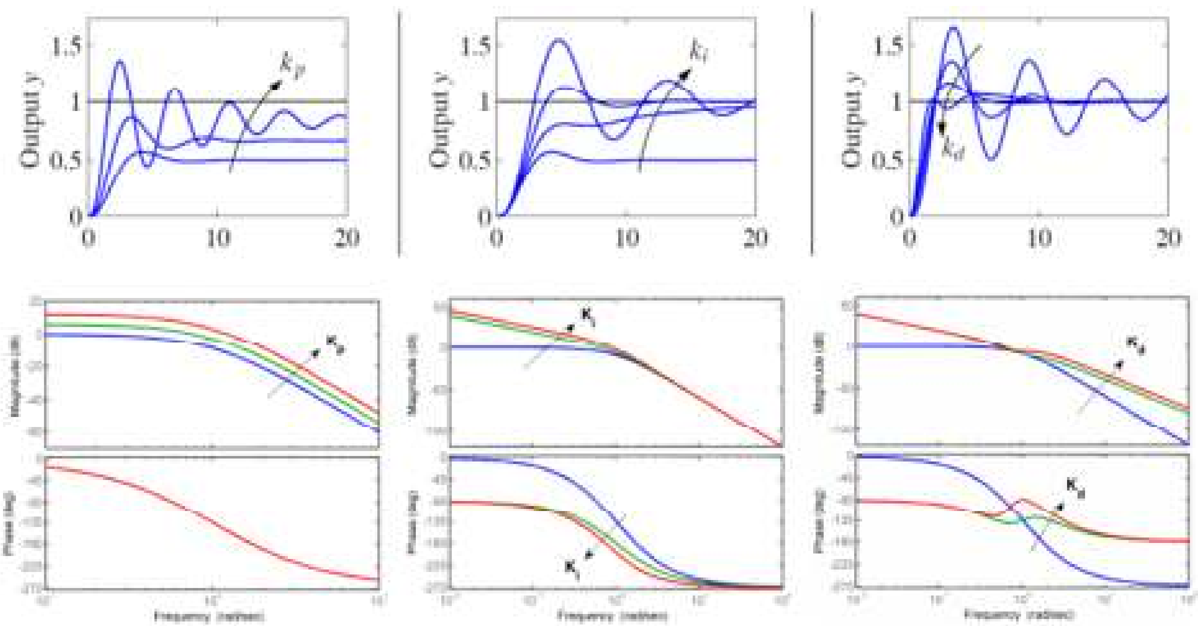
Or in Laplace domain:

$$U(s) = K_p E(s) + K_i \frac{1}{s} E(s) + K_d s E(s) = (K_p + \frac{K_i}{s} + K_d s)E(s)$$

PD and PI controllers are analogous to lead and lag compensators:

- PID controller → Lead-lag compensator

PD-Lead Compensator	PI–Lag Compensator
PD controller format: $$G_b(s) = \frac{K_1 s + K_2}{\tau_d s + 1}$$ Lead compensator format: $$G_{ld}(s) = K_{ld}\frac{s + 1/T}{s + \frac{1}{\alpha T}}$$	The following compensator can be seen to contain P and I terms when divided into components: $$G_{PI} = \frac{K_1 s + K_2}{s} = K_1 + \frac{K_2}{s}$$ This is similar to the format of a lag compensator (with $\beta \to \infty$): $$G_{lg}(s) = K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\beta T}}$$



P PI PID

Discrete and Continuous Time

Continuous Time	Discrete Time
Dynamics are described by **differential equations** Current slope (derivative) determined by current state and input. $\dot{x} = Ax + u$	Dynamics are described by **difference equations**. Next state are determined by previous state and input $x(k+1) = Ax(k) + u(k)$

Z-Transform

The Z-transform converts a discrete-time signal into a **complex, frequency-domain** representation**.**

- Discrete-time equivalent of **Laplace** transform.
- Z is a complex variable

$$kh = t_k \text{ and } y(k) = y(t_k)$$

Consider the discrete-time signal {f(k): k=0,1,...}. The z-transform of $f(kh)$ is defined as

$$\mathcal{Z}\{f(k)\} = F(z) = \sum_{k=0}^{\infty} f(k) z^{-k}, \quad f(k) = 0 \text{ if } k < 0$$

__Examples__

With initial conditions as zero (used when taking Z-transform to find transfer function):

$$y(k+2) \to z^2 Y(z) \qquad 5y(k+1) \to 5zY(z) \qquad 8y(k) \to 8Y(z)$$

__Generalised__

$$\mathcal{Z}\{f(k-i)\} = z^{-i} F(z) \qquad \mathcal{Z}\{f(k+i)\} = z^i F(z) - \sum_{j=1}^{i} z^j f(i-j)$$

Initial value theorem

$$\lim_{k \to 0} f(k) = \lim_{z \to \infty} zF(z)$$

Final value theorem

$$\lim_{k \to \infty} f(k) = \lim_{z \to 1} (z-1) F(z)$$

Pulse Transfer Function

From the general **difference equation**:

$$y(k+n) + a_1 y(k+n-1) + \cdots + a_n y(k) = b_1 u(k+n-1) + \cdots + b_n u(k)$$

Taking the z-transform produces the **pulse transfer function**:

$$(z^n + a_1 z^{n-1} + \cdots + a_n) Y(z) = (b_1 z^{n-1} + \ldots + b_n) U(z) \qquad H(z) = \frac{b_1 z^{n-1} + \cdots + b_n}{z^n + a_1 z^{n-1} + \cdots + a_n}$$