Econometric Principles ECON232

Probability

Definitions

Random Experiment:
A process leading to at least two possible outcomes with uncertainty as to which will occur.

- Toss two coins.
- Access the ASX website tomorrow morning and observe the opening price of NAB.

Sample Space (or Population):
The set of all possible outcomes of a random experiment.

- Toss 2 coins. The sample space is:

$$\Omega = \{\{H, H\}, \{H, T\}, \{T, H\}, \{T, T\}\}$$

- Ω = The set of non-negative real numbers (i.e. all the values that the NAB opening price could take).

Event:
A subset of the sample space.

- Toss 2 coins. Define the event

$$A = \{\{H, T\}, \{T, H\}\}$$

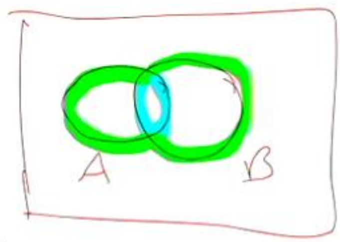
- Let B be the event that the opening price of NAB is less than \$20.

Some set notation:

- $A \cup B$ denotes the union of sets A and B.
- $A \cap B$ denotes the intersection of sets A and B.
- $A \subset \Omega$ denotes that set A is a subset of set Ω.

$A \cup B$

$A \cap B$



$A \subset \Omega$

Definition of Probability

For every event A in the sample space, the probability P is a function of A for which

1) $P(A) > 0$.
2) $P(\Omega) = 1$.
3) For mutually exclusive events A and B,
   $P(A \cup B) = P(A) + P(B)$.

Conditional Probability:

The probability of event A conditional on event B is defined as:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Helps to shrink down the sample space. Eg: unemployment rate

Statistical Independence:

Two events, A and B are said to be statistically independent if

$$P(A \cap B) = P(A)P(B)$$

Note: It follows that, if A and B are independent, then

$$P(A|B) = P(A)$$

Random Variables

A **random variable** is a function which assigns a number to each element of the sample space Ω.

- Toss a coin and define the variable

$$X = \begin{cases} 0 \text{ if head} \\ 1 \text{ if tail} \end{cases}$$

- Roll a die and define the variable

X = the number on the upper face

A **discrete random variable** is a random variable whose set of possible values is enumerable. Eg: see previous two examples.

A **continuous random variable** is a random variable whose set of possible values is not enumerable. E.g. repeatedly roll a die until the outcomes form a valid student ID number. Find that student and measure his/her height. Define the variable X = height

Probability Density Functions (PDF's)

A PDF of a **discrete** random variable X (which takes one of m possible values) is a function which associates each possible value of X with its probability. i.e.

$$f_X(x_i) = \begin{cases} P(X = x_i) \text{ for } i = 1, 2, ..., m \\ 0 \text{ otherwise} \end{cases}$$

Properties:

1) $f_X(x_i) \geqslant 0$ for $i = 1, 2, ..., m$.
2) $\sum_{i=1}^{m} f_X(x_i) = 1$.

Example

Toss a coin twice and define the random variable

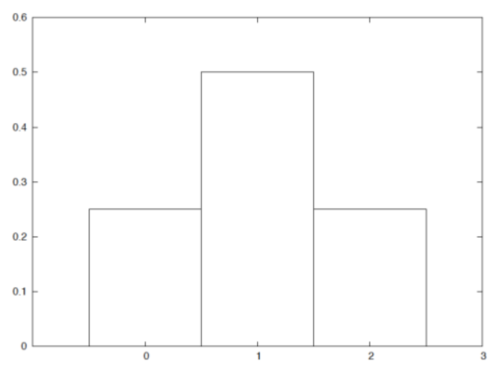
$$X = \begin{cases} 0 \text{ if } \{H,H\} \\ 1 \text{ if } \{T,H\} \text{ or } \{H,T\} \\ 2 \text{ if } \{T,T\} \end{cases}$$

Table: PDF of X

x	$f_X(x)$
0	0.25
1	0.5
2	0.25

Example

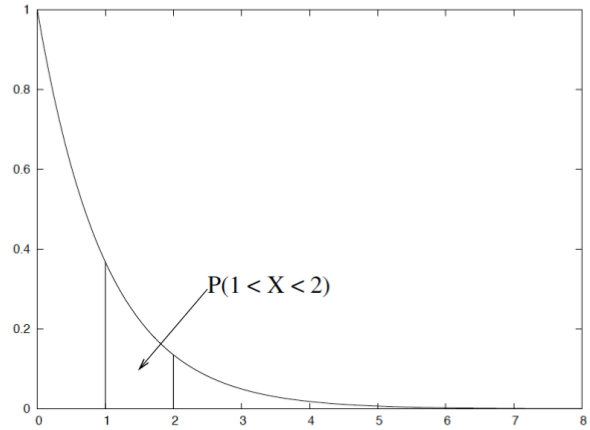
Figure: PDF of X



A PDF of a **continuous** random variable X is a function $f_X(x)$ such that the area under the curve between any two points a and b is equal to P(a < X < b).

e.g. $f_X(x) = e^{-x} \quad x \geqslant 0$

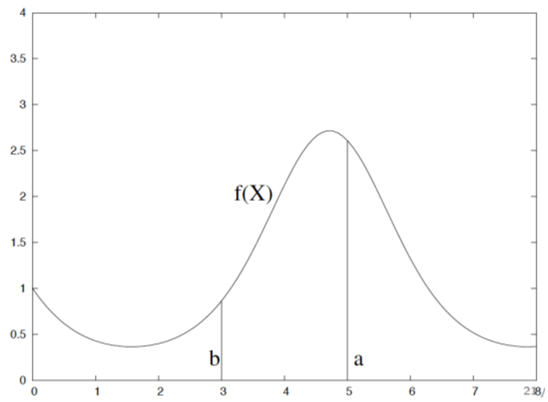
Figure: PDF of X



Notation: $\int_b^a f(x)dx$

The notation means the area bordered by the horizontal axis, the function f(x), the upper bound a and the lower bound b. This area is called the **integral** of f(x) between a and b.

e.g. $\int_b^a f(x)dx$



The area under the curve is its probability

Therefore, the continuous PDF of a random variable X may be defined as the function $f_X(x)$ for which:

$$\int_b^a f_X(x)dx = P(b < X < a)$$

Properties:

1) $f_X(x) \geqslant 0$ for $-\infty < x < \infty$.
2) $\int_{-\infty}^{\infty} f_X(x)dx = 1$.

Expected Value

The expected value of a discrete random variable X is defined as:

$$E(X) = \sum_{i=1}^{N} x_i f_X(x_i)$$

The expected value of a continuous random variable X is defined as:

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx$$

The expected value of X is referred to as the mean of X and is denoted μ_X

Example of Expected Value

Table: PDF of X

x	$f_X(x)$
0	0.25
1	0.5
2	0.25

$$E(X) = \sum_{i=1}^{N} x_i f_X(x_i) = 0 \times 0.25 + 1 \times 0.5 + 2 \times 0.25 = 1$$

Properties of the Expected Value:

If X and Y are random variables and is a constant, then:

- $E(\alpha X) = \alpha E(X)$.
- If g(X) is a continuous function of a discrete random variable X, then

$$E\left(g(X)\right) = \sum_{i=1}^{N} g(x_i) f_X(x_i)$$

- If g(X) is a continuous function of a continuous random variable X, then

$$E\left(g(X)\right) = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

Variance

If X is a random variable, then the variance of X is:

$$Var(X) = \sigma_X^2 = E\left((X - E(X))^2\right)$$
$$= E(X^2) - E(X)^2$$

σ_x is referred to as the standard deviation

Properties of the Variance:

If X is a random variable and a constant, then:

- $Var(X + \alpha) = \sigma_X^2$
- $Var(\alpha X) = \alpha^2 \sigma_X^2$.

Example of Variance

Table: PDF of X

x	$f_X(x)$
0	0.25
1	0.5
2	0.25

$$E(X^2) = \sum_{i=1}^{N} x_i^2 f_X(x_i) = 0^2 \times 0.25 + 1^2 \times 0.5 + 2^2 \times 0.25 = 1.5$$

$$Var(X) = E(X^2) - E(X)^2 = 1.5 - 1^2 = 0.5$$

Skewness and Kurtosis

For a random variable X, the skewness is defined as:

$$S = \frac{E(X-\mu)^3}{\sigma^3}$$

- If S > 0 then the PDF is right-skewed.
- If S < 0 then the PDF is left-skewed.
- If S = 0 then the PDF is symmetric.

For a random variable X, the kurtosis is defined as

$$K = \frac{E(X-\mu)^4}{\sigma^4}$$

- If K < 3 then the PDF is platykurtic (fat, short-tailed).
- If K > 3 then the PDF is leptokurtic (thin, long-tailed).
- If K = 3 then the PDF is mesokurtic.

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Moments:

$E(X^k)$ is called the k^{th} moment of X.
$E((X - \mu)^k)$ is called the k^{th} centred moment of X

Cumulative Distribution Functions (CDF's)

For a random variable X, the CDF is defined as:

$$F_X(x) = P(X\ 6 \leq x)$$

If X is discrete, then:

$$F_X(x_i) = \sum_{j=1}^{i} f_X(x_j) \text{ for } i = 1, \ldots, m$$

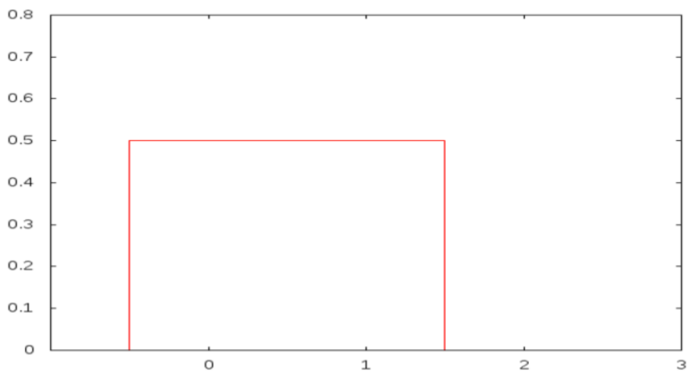
If X is continuous, then:

$$F_X(x) = \int_{-\infty}^{x} f_X(z)dz$$

Distributions

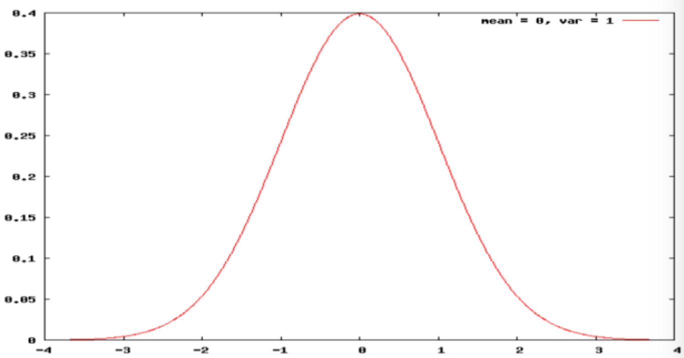
Uniform Distribution

$$f_X(x) = \begin{cases} \frac{1}{a-b} \text{ for } b < X < a \\ 0 \text{ otherwise} \end{cases}$$



Gaussian (Normal) Distribution

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



Interesting Properties of The Gaussian (Normal) Distribution:

If X and Y are each normally distributed and is a constant, then:

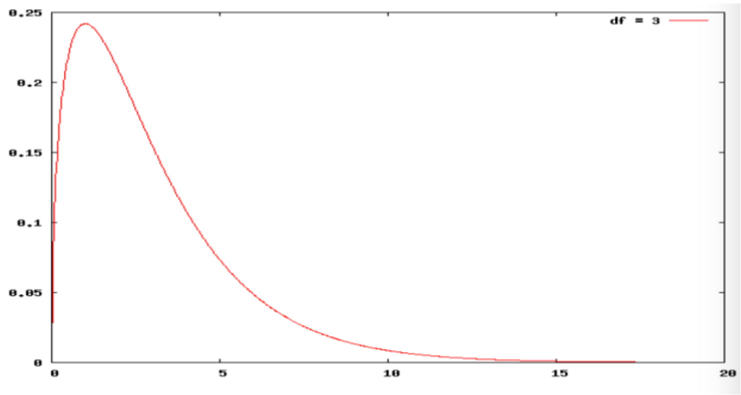
(i) αX is normally distributed.
(ii) $X - \alpha$ is normally distributed.
(iii) $X + Y$ is normally distributed.

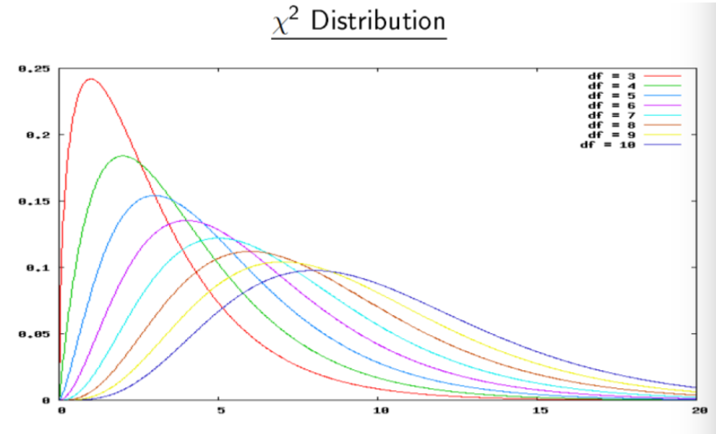
χ^2 Distribution

Let $X_1, X_2 \ldots X_N$ be a sequence of independent, zero-mean, unit-variance, Gaussian random variables.

$$\text{Let } Y = \sum_{i=1}^{N} X_i^2.$$

Then Y is said to have a χ^2 distribution with N degrees of freedom. We write this as $Y \sim \chi^2_N$



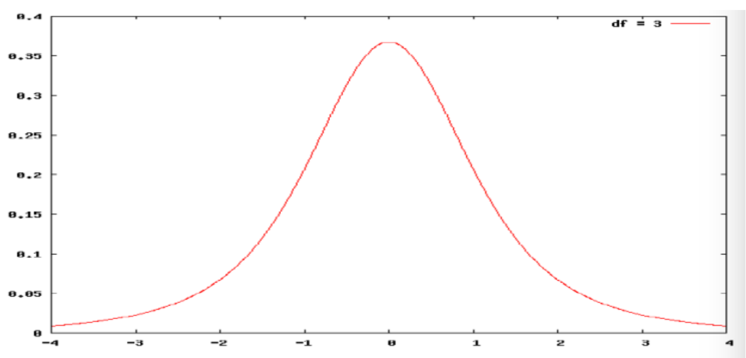


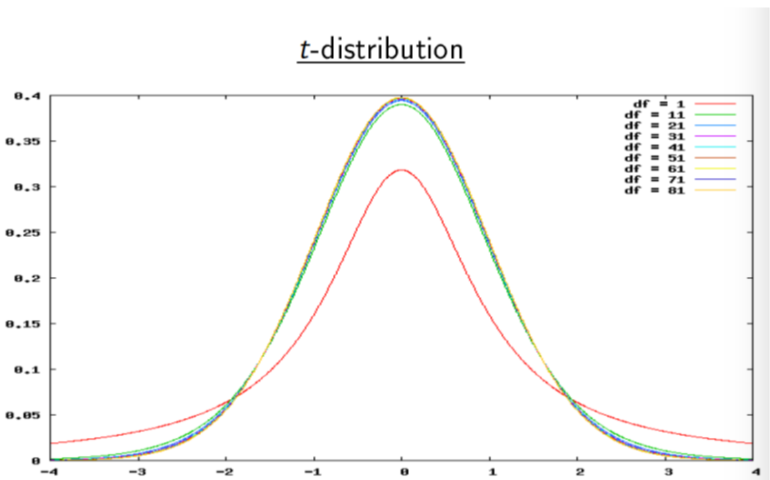
T-Distribution

Let $X \sim N(0, 1)$ and $Y_N \sim X^2_N$ where X and Y_N are independent. The distribution of:

$$t = \frac{X}{\sqrt{Y_N/N}}$$

is called the (Student's) t-distribution with N degrees of freedom. We write this as $t \sim t_N$





F-Distribution

Let $X_m \sim \chi^2_m$ and $Y_N \sim \chi^2_N$ where X_m and Y_N are independent. The distribution of:

$$F = \frac{X_m/m}{Y_N/N}$$

is called the F-distribution with numerator degrees of freedom m and denominator degrees of freedom N. We write this as $F \sim F_{m,N}$