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Lecture 1. Tuesday, 1 August 2017
- Nathan Duignan (PhD student); Holger Dullin away 1st week

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Lecture notes are essential and can be purchased from KOPYSTOP.

Tutorials

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Introduction

Newtownian mechanics:
- 3 laws of motion
- Based off calculus

Analytical mechanics
- Based off principle of least action
  - From lagrangian and Hamiltonian mechanics
- Energy becomes fundamental quantity of study
- System moves to the minimum energy state of the system
- Based off Calculus of Variations
Chapter 1: Calculus of Variations

Calculus of variations is a field of mathematical analysis that deals with maximizing or minimizing functionals, which are mappings from a set of functions to the real numbers. Functionals are often expressed as definite integrals involving functions and their derivatives. The interest is in extremal functions that make the functional attain a maximum or minimum value – or stationary functions – those where the rate of change of the functional is zero.

A simple example of such a problem is to find the curve of shortest length connecting two points. If there are no constraints, the solution is obviously a straight line between the points. However, if the curve is constrained to lie on a surface in space, then the solution is less obvious, and possibly many solutions may exist. Such solutions are known as geodesics. A related problem is posed by Fermat’s principle: light follows the path of shortest optical length connecting two points, where the optical length depends upon the material of the medium. One corresponding concept in mechanics is the principle of least action.

Many important problems involve functions of several variables. Solutions of boundary value problems for the Laplace equation satisfy the Dirichlet principle. Plateau’s problem requires finding a surface of minimal area that spans a given contour in space: a solution can often be found by dipping a frame in a solution of soap suds. Although such experiments are relatively easy to perform, their mathematical interpretation is far from simple: there may be more than one locally minimizing surface, and they may have non-trivial topology.

Basic idea of calculus of variations

Regular calculus:
- Find some \( x \in \mathbb{R} \) that minimises the function \( f(x) \)
  \[ f'(x) = 0; \quad f''(x) > 0 \]

Calculus of Variations
- Find a function \( y(x) \) that minimises the functional

Functional:

In mathematics, and particularly in functional analysis and the calculus of variations, a functional is a function from a vector space into its underlying field of scalars. Commonly the vector space is a space of functions; thus the functional takes a function for its input argument, then it is sometimes considered a function of a function (a higher-order function). Its use originates in the calculus of variations, where one searches for a function that minimizes a given functional. A particularly important application in physics is searching for a state of a system that minimizes the energy functional.

- A mapping that from a function to a real number
  \[ f \to f(x_0) \]

Functional notation:

\[ L[y] = \int_{x_0}^{x_1} F(x, y(x), y'(x)) \, dx \]
Calculus of variations finds minimises this functional

**Classic problems:**

1. **Geodesics (Riemann ~ 1854)**
   
   We want shortest curve joining two points
   
   \[ L = \int_A^B ds \]
   
   Using pythagorus:
   
   \[ ds^2 = dx^2 + dy^2 \]
   
   - This is called the “metric” (you can look at other surfaces and Riemann spaces)
   
   We “can” factor out \( dx \); to get that:
   
   \[ ds^2 = \left( 1 + \left( \frac{dy}{dx} \right)^2 \right) dx^2 \]
   
   \[ \Rightarrow L = \int_{x_A}^{x_B} \sqrt{1 + (y')^2} dx \]
   
   Or, if parametrically (if \( y \) is not a function of \( x \))
   
   \[ L = \int_{t_A}^{t_B} \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} dt \]

2. **Catenoide (Euler ~ 1744)**
   
   - Soap film
     
     - Minimise the potential energy of the soap film between 2 rings
- We have 2 circles of different radius; $a$ and $b$. Taking small cuts of surface area

\[ \text{SA} = \int_A^B 2\pi y \, ds \]

\[ = \int_0^L 2\pi y \sqrt{1 + (y')^2} \, dx \]

3. Brachistochrone (John Bernoulli ~ 1696)
Path of shortest time from $A \rightarrow B$ (falling under gravity)
We want the kinetic energy at $B$ to be equal to the gravitational potential energy at $A$

$$KE = PE$$
$$\frac{1}{2}mv^2 = mg(h - y)$$

the time taken will be: $dt = \frac{ds}{v}$

$$T = \int_A^B \frac{ds}{v}$$
4. **Queen Dido’s city (isoperimetric problem)**
Want shape with maximum area with a fixed perimeter \( L \)

\[
A = \int_0^a y \, dx
\]

- But now we don’t have the constraint that we have a fixed perimeter \( L \).
  - To do this, we rewrite this in terms of \( ds \)

\[
A = \int_0^L y \left( \frac{dx}{ds} \right) ds
\]

We use that \( ds^2 = dy^2 + dx^2 \)

\[
\Rightarrow \left( \frac{dx}{ds} \right)^2 + \left( \frac{dy}{ds} \right)^2 = 1
\]

\[
\therefore A = \int_0^L y \sqrt{1 - y'^2} \, ds
\]

5. **Catenary (Huygens~ 1691)**
Curve of a hanging chain between 2 points?
Fixed length $L$

We want to minimise the potential energy of the chain, with constant density $\rho$

$$V = \int_A^B \rho g y ds$$

- Again we don’t have the constraint of the length $L$

To do this, we will use a lagrange multiplier; to force the constraint

$$V + \lambda L = \int_A^B (\rho g y + \lambda) ds$$

Lecture 2.

Lagrange equations

In the calculus of variations, the Euler–Lagrange equation, Euler’s equation, or Lagrange’s equation (although the latter name is ambiguous—see disambiguation page), is a second-order partial differential equation whose solutions are the functions for which a given functional is stationary. It was developed by Swiss mathematician Leonhard Euler and Italian-French mathematician Joseph-Louis Lagrange in the 1750s.

Because a differentiable functional is stationary at its local maxima and minima, the Euler–Lagrange equation is useful for solving optimization problems in which, given some functional, one seeks the function minimizing or maximizing it. This is analogous to Fermat’s theorem in calculus, stating that at any point where a differentiable function attains a local extremum its derivative is zero.

In Lagrangian mechanics, because of Hamilton’s principle of stationary action, the evolution of a physical system is described by the solutions to the Euler–Lagrange equation for the action of the system. In classical mechanics, it is equivalent to Newton’s laws of motion, but it has the advantage that it takes the same form in any system of generalized coordinates, and it is better
suited to generalizations. In classical field theory there is an analogous equation to calculate the dynamics of a field.

In each above example we have an integral containing a function and a derivative, we will look at the conditions and how to solve these.

One dependent variable
Consider functional

\[ I[y] = \int_a^b F(x, y, y') \, dx \]

where \( F \) is an arbitrary function of independent \( x \) and dependent \( y \), and its derivative. Let \( y(a) = c, y(b) = d \).

Suppose \( y = f(x) \) is the desired function that causes \( I \) to have an extreme value. Consider the class of admissible functions

\[
\begin{align*}
y(x) &= f(x) + \epsilon \eta(x) \\
y'(x) &= f'(x) + \epsilon \eta'(x)
\end{align*}
\]

Where \( \eta \) is arbitrary and differentiable. \( \eta(a) = \eta(b) = 0 \) to satisfy the endpoint conditions. \( \epsilon \ll 1 \) so that \( y \) lies close to \( f(x) \).

Substitute the form of the admissible function into the integral and consider it as a function of \( \epsilon \)

\[
I[f + \epsilon \eta] = \int_a^b F(x, f + \epsilon \eta, f' + \epsilon \eta') \, dx
\]

If \( \eta \) is considered fixed, the extremum is when \( \frac{dI}{d\epsilon} = 0 \), so we require

\[
\left. \frac{d}{d\epsilon} I[f + \epsilon \eta] \right|_{\epsilon = 0} = 0
\]

Differentiating \( I \):

\[
\frac{d}{d\epsilon} I[f + \epsilon \eta] = \int_a^b \frac{dF}{d\epsilon} \, dx = \int_a^b \left\{ \frac{\partial F}{\partial y} \frac{dy}{d\epsilon} + \frac{\partial F}{\partial y'} \frac{dy'}{d\epsilon} \right\} \, dx = \int_a^b \left\{ \frac{\partial F}{\partial y} \eta + \frac{\partial F}{\partial y'} \eta' \right\} \, dx
\]

Integrating by parts:

\[
= \int_a^b \left\{ \frac{\partial F}{\partial y} \eta(x) - \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) \eta(x) \right\} \, dx + \frac{\partial F}{\partial y'} \eta(x) \bigg|_a^b
\]

- Note that IBP uses the total derivative.

As \( \eta(a) = \eta(b) = 0 \), we get that

\[
\int_a^b \left\{ \frac{\partial F}{\partial y} \eta(x) - \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) \eta(x) \right\} \, dx = 0
\]
As this is true \( \forall \eta \), we must have that
\[
\frac{\partial F}{\partial y} = \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right)
\]

\textit{Euler-Lagrange Equation}

\[
\frac{\partial F}{\partial y} = \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right)
\]

- Note that \( \frac{d}{dx} \) is the total derivative
- Thus, imposing this condition insures that \( y = f(x) \) minimises the integral of \( I[y] \).
- This brings us the theorem:

\textbf{Theorem 1:}

Assume \( F(x,y,y') \) is twice differentiable in each argument. Then any function \( y(x) \) that satisfies the fixed end point boundary condition \( y(a) = c, y(b) = d \) is twice differentiable in the closed interval \( x \in [a, b] \) and that minimises the functional

\[
I[y] = \int_a^b F(x,y,y') dx
\]

Satisfies the Euler-Lagrange equations

\[
\frac{\partial F}{\partial y} = \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right)
\]

\textbf{Notes:}

- Any solution of the EL equation that satisfies the BC is a critical point
- May minimise, maximise or saddle point
- The equation above is the total derivative in \( (x, y, y') \); in full the equation is

\[
\frac{\partial F}{\partial y} = \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial y'} \right) + \frac{\partial}{\partial y} \left( \frac{\partial F}{\partial y'} \right) y' + \frac{\partial}{\partial y'} \left( \frac{\partial F}{\partial y'} \right) y''
\]

So that the EL equation is second order ODE as long as \( \frac{\partial^2 F}{\partial y'^2} \neq 0 \)

\textbf{Example: Geodesic}

We had \( F = \sqrt{1 + y'^2} \)

EL equation becomes

\[
\frac{\partial F}{\partial y} = \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right)
\]

\[
\therefore 0 = \frac{d}{dx} \left( \frac{y'}{\sqrt{1 + y'^2}} \right) = \frac{y''}{\sqrt{1 + y'^2}} - \frac{y'^2 y''}{(1 + y'^2)^{3/2}} = \frac{y''}{\left( \frac{1}{\sqrt{1 + y'^2}} \right)^{3/2}}
\]
As the denominator cannot vanish, this is equivalent to \( y'' = 0 \); so that we get a straight line (what we expect for the shortest distance between points on a flat plane).

Example: catenoid soap film

\[
F = y\sqrt{1 + y'^2}
\]

\[
\frac{\partial F}{\partial y} = \frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right)
\]

\[
\therefore \sqrt{1 + y'^2} = \frac{d}{dx} \left( \frac{yy'}{\sqrt{1 + y'^2}} \right)
\]

Simplifying to

\[
yy'' = (1 + y'^2)
\]

Solution is a cosh function.

More than 1 dependent variable.

Consider the case of more than 1 dependent variable, where we define the dependent variables parametrically by the parameter \( t \); so that

\[
S = \int_{t_1}^{t_2} L(t, q_1, ..., q_n, \dot{q}_1, ..., \dot{q}_n) dt
\]

With \( n \) variables of \( q_i \) and their derivatives \( \dot{q}_i \)

Using the same idea as before, the lagrange equations are satisfied; using \( \vec{q} \) to denote the extrema

\[
q_i = \vec{q}_i + \epsilon_i \eta_i
\]

Using the same process, the necessary conditions becomes

\[
\frac{\partial S}{\partial \epsilon_1} \bigg|_{\epsilon_1=...=\epsilon_n = 0}, ..., \frac{\partial S}{\partial \epsilon_n} \bigg|_{\epsilon_1=...=\epsilon_n = 0} = 0
\]

Simultaneously for each DOF, so we get \( \forall i \) DOF that

\[
\frac{\partial L}{\partial q_i} = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right)
\]

- Note again the total derivative

We can also expand the total derivatives to get

\[
\frac{\partial L}{\partial q_i} = \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{q}_i} \right) + \sum_{j=1}^{n} \frac{\partial}{\partial q_j} \left( \frac{\partial L}{\partial \dot{q}_i} \right) \dot{q}_j + \sum_{j=1}^{n} \frac{\partial}{\partial \dot{q}_j} \left( \frac{\partial L}{\partial q_i} \right) \dot{q}_j
\]

For each degree of freedom \( i \)

- The system has a general solution containing exactly the correct number of arbitrary constants provided that
\[
\det \left( \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \right) \neq 0
\]

- This is singular if this is satisfied. We will only look at a few nonsingular examples in this course.

**First integrals**

- Solving DEs is similar to integration, we can replace a DE of a higher order with an integral of a lower order (which will becomes a DE of lower order)

\[
\frac{\partial L}{\partial q_i} = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right)
\]

Looking at the EL equations: the LHS is partial derivative, while the RHS is total.

**L independent of \( q_i \)**

If \( L \) does not depend on \( q_i \) then \( \frac{\partial L}{\partial q_i} = 0 \), giving that \( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = 0 \); so that the partial derivative must be a constant

\[
\frac{\partial L}{\partial \dot{q}_i} = C_i
\]

**L independent of \( t \) (quasitonomous)**

Now:

Calculate the total derivative of \( L \) wrt \( t \)

\[
\frac{dL}{dt} = \frac{\partial L}{\partial t} + \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i + \sum_{j=1}^{n} \frac{\partial L}{\partial q_i} \ddot{q}_i
\]

Using the EL in the first sum

\[
= \frac{\partial L}{\partial t} + \sum_{j=1}^{n} d \left( \frac{\partial L}{\partial \dot{q}_i} \right) \dot{q}_i + \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i
\]

Combining the two sums and noticing total derivative are the product rule

\[
= \frac{\partial L}{\partial t} + \sum_{j=1}^{n} d \left( \frac{\partial L}{\partial \dot{q}_i} \right) \dot{q}_i
\]

Rearranging:

\[
\frac{\partial L}{\partial t} = \frac{d}{dt} \left( L - \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right)
\]

As LHS is partial and RHS is total derivatives; we can use the same logic as before so that, if \( L \) is independent of \( t \)

\[
\sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L = C
\]
A constant.

**Example: geodesic**

\[ F(x, y, y') = \sqrt{1 + y'^2} \]

As the integrand does not depend on \( y \), we get a first integral

\[ \frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}} = C \]

\[ \rightarrow y' = C \]

(another straight line)

**Example: brachistrochrone:**

\[ F(y, y') = \sqrt{\frac{1 + y'^2}{h - y}} \]

Giving the 1st integral

\[ y' \left( \frac{y'}{\sqrt{(h - y)(1 + y'^2)}} \right) - \frac{1 + y'^2}{\sqrt{h - y}} = C \]

\[ \rightarrow -1 = C \sqrt{h - y} \sqrt{1 + y'^2} \]

\[ y' = \pm \sqrt{\frac{\alpha^2 - h + y}{h - y}} \]

Where \( \alpha = \frac{1}{C} \)

Separating and integrating:

\[ \int \frac{\sqrt{h - y}}{\sqrt{\alpha^2 - h + y}} dy = \pm \int dx = \pm x + D \]

We can use \( h - y = \alpha^2 \sin^2 \theta \); to find the overall solution

\[ x = \alpha^2 \left( \theta - \frac{1}{2} \sin 2\theta \right) - D \]

\[ y = h - \alpha^2 \sin^2 \theta \]

As the parametric representation.

---

Lecture 3.

**Exact derivatives**

Suppose \( F(x, y, y') \) can be written as a total derivative of some function of the dependent and independent variables

\[ F(x, y, y') = \frac{d}{dx} G(x, y) \]

- Note no derivative
The integral can now be evaluated explicitly
\[
\int_a^b F(x, y, y') \, dx = \int_a^b \frac{dG}{dx} \, dx = G(b, y_b) - G(a, y_a)
\]

And it depends only on the **end points**, and not the function \( y \) inbetween. So every function satisfying the end point conditions will give the same value.

Looking at the EL equation and expanding using the chain rule
\[
F(x, y, y') = \frac{d}{dx} G(x, y) = \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} \frac{dy}{dx} + \frac{\partial G}{\partial y'} y'
\]

Now
\[
\frac{\partial F}{\partial y} = \frac{\partial^2 G}{\partial y \partial x} + \frac{\partial^2 G}{\partial y^2} \frac{dy}{dx}
\]
\[
\frac{\partial F}{\partial y'} = \frac{\partial G}{\partial y} \quad \text{[no } y' \text{ dependence of } G]\]
\[
\frac{d}{dx} \left( \frac{\partial F}{\partial y'} \right) = \frac{d}{dx} \left( \frac{\partial G}{\partial y} \right) = \frac{\partial^2 G}{\partial x \partial y} + \frac{\partial^2 G}{\partial y^2} \frac{dy}{dx}
\]

So the EL equation reduce to
\[
\frac{\partial^2 G}{\partial y \partial x} + \frac{\partial^2 G}{\partial y^2} \frac{dy}{dx} = \frac{\partial^2 G}{\partial x \partial y} + \frac{\partial^2 G}{\partial y^2} \frac{dy}{dx}
\]

Simplifying
\[
\frac{\partial^2 G}{\partial y \partial x} = \frac{\partial^2 G}{\partial x \partial y}
\]

Which is trivially satisfied.

Lecture 4. Tuesday, 8 August 2017

**Last week:**
In lagrangian dynamics: we want to extremise
\[
S[q] = \int L(t, q_1, ..., q_n, \dot{q}_1, ..., \dot{q}_n) \, dt
\]

\[\delta S = 0\]

- Hamilton’s principle

To minimise this, we use the ELE
\[
\frac{\partial L}{\partial q_i} = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right)
\]
In almost all cases, these DEs cannot be solved, but we can use conservation laws to help simplify/solve them.

Recall: first integral

1. If $L$ is independent of $q_i$, then $\frac{\partial L}{\partial q_i} = 0$

   $\therefore \frac{\partial L}{\partial q_i} = C$

2. If $L$ is independent of $t$ (autonomous); then

   $\sum_{i=1}^{n} q_i \left( \frac{\partial L}{\partial q_i} \right) - L = C$

ELE are unchanged:

- $L \rightarrow L$ is constant
- $L \rightarrow L + \frac{d}{dt} G(t, q_1, ..., q_n)$ for arbitrary $G$ is particular

$L \rightarrow L + C$
$L \rightarrow L + f(t)$

Chapter 2: Lagrangian dynamics

History:

What is the function $L$?

- Newton: static equilibrium at minimum of potential energy $V(q)$: $\delta \int V(q) dt = 0$
- Euler/Maupertuis: assume constant energy (potential and kinetic); $T + V = const \Rightarrow \delta \int T(q, \dot{q}) dt = 0$

\[
T = \frac{1}{2} mv^2; V = \frac{ds}{dt}
\]

- The “action” is then

\[
\int T dt = \int \frac{m}{2} \left( \frac{ds}{dt} \right)^2 dt = \frac{m}{2} V ds
\]

- Minimising this is then “principle of least action”

- Lagrange:

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}} \right) = \frac{\partial T}{\partial q} - \frac{\partial V(q)}{\partial q}
\]

- Poisson:

\[
L = T - V \qquad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q}
\]
With $T = T(q, \dot{q}); V = V(q)$

So if we have no time dependence, we get the conservation of energy that

$$T + V = Const$$

**Configuration Space and state Space**

The **configuration** of a system of particles is said to be known if the **position** of each particle in the system is known.

**Definitions:**

- If we have $n$ degrees of freedom
  
  \{q_1, ..., q_n\} is the **configuration space**
  
  \{q_1, ..., q_n, \dot{q}_1, ..., \dot{q}_n\} is the **phase space (state space)**

**Common types of potentials**

**Free particle ($V = 0$)**

- A free particle does not experience any force or interactions, and so $V = 0$

If $V = 0$

1 particle in 3 dim space: 3DOF; 2D space: 2DOF

$n = 1$ DOF

$$L = T = \frac{m}{2}v^2 = \frac{m}{2}\left(\frac{ds}{dt}\right)^2$$

**Cartesian coordinates**

In Cartesian coordinates:

$$ds^2 = dx^2 + dy^2 + dz^2$$

Giving that

$$\left(\frac{ds}{dt}\right)^2 = v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

So we get the lagrangian:

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

We get the ELE for $x$

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$$

$$\Rightarrow 0 = \frac{d}{dt}(m\ddot{x}) = m\dddot{x}$$

For each $x, y, z$

$$m\dddot{x} = 0$$

$$\therefore m\dddot{x} = C$$

Or