AMME3060 ENGINEERING METHODS

LECTURE NOTES

Contents

AMME3060 Engineering Methods Lecture Notes

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Introduction:

Objective:

- Understand numerical solution methods which can be used to solve engineering problems in heat transfer, fluids and solids and the implementation of these methods in commercial packages.
- Understand the mathematical basis of numerical solution methods:
  - Ø finite element, finite difference and finite volume methods.
  - Ø direct, iterative linear solvers and non-linear solvers.
  - Ø mesh generation and best practice in commercial packages
  - Ø numerical stability
- By the end of this course
  - Ø you will be able to approach any commercial engineering software package and have a deep understanding of how to different settings within a package will effect computational efficiency, stability and accuracy.
  - Ø You will have expert level competency in the use of computational software such as ANSYS.

Approach

- Understand Mathematical methods and concepts
  - Lectures, reading of lecture notes.
  - Tutorial practice problems and assignments

- Link the theory to practice: Demonstrate numerical accuracy, stability and efficiency and understand implementation
  - Use a simple Android App to test concepts
  - Write simple Matlab scripts and test accuracy, speed and stability.
  - Use commercial packages to test the same problem and realise the same behavior exists in commercial packages.

- Expert use of Commercial packages
  - You will practice these ANSYS skill in PC laboratories and in Assignments.

- Understand the range of methods which can be used to solve complex engineering problems in heat transfer, fluids and solids.
  - Range of problems used throughout the course.

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Week #	Due	Lab: MON/TUE	Lecture 1: WED	Lecture 2: THUR
1			Trial Functions	Heat Equation
2		MATLAB: ERROR	Weighted Residuals	FEM: Galerkin
3		ANSYS 1: INTRO.	FEM: Galerkin	FEM: Quadratic
4			FEM: 2D	FEM: 2D
5		ANSYS 2: MESHING 1	Mesh Generation	Mesh Generation
6	**Assign 1.**	ANSYS 3: MESHING 2	FDM	FDM/FVM
7			Solvers	Solvers
8	**QUIZ 1**	ANSYS 4: UNSTEADY	**QUIZ 1**	Unsteady FDM
BREAK				
9		*Holiday*	Unsteady FDM	Unsteady FEM
10			Stability	Stability
11	**Assign. 2**		CFD	CFD
12	**Quiz 2**	ANSYS 5: CFD	**QUIZ 2**	**Guest Lecture (compulsory)**
13			Non-Linear Solvers	**Standards (compulsory)**

- Quiz in lectures

Assessment

Quizzes (10% x 2)

- 2 Quizzes will be held, each worth 10% of the course assessment
  - Held in the Wednesday Lecture on WEEK 8 and 12. Closed book. Some students will be directed to sit their quiz in ABS Seminar Room 2060. This will be indicated by an student Id range to be announced later.
- Exam* (50%)
  - 2 hours held in the Exam period

  - *Require 50% in exam to pass the course.
- Assignment (12% x 2)
  - Assignments will involve writing Matlab code to solve steady and unsteady problems. These will focus on heat transfer problems, but similar equations are applicable to stress analysis and many types of engineering analysis.
  - You will also use the ANSYS thermal solver package to solve assignment the problems.
  - You will learn the concepts in the lectures/tutorials, then get some initial assistance with coding in the tutorials and learn ANSYS both in the lab sessions and in self guided tutorials.
  - The assignments will be difficult and lengthy. Manage your time effectively.
  - Submit the assignments through Blackboard by 5pm on Friday of the week due. Late penalties of 25% per day apply

Laboratories

- Labs will be held on Weeks 2,3, 5,6,8 and 12. Ignore other timetabled lab slots.
  - Each lab has a weighting of 1%. The marking breakdown:
    - Ø 0.5% for completion of all the lab tasks and
    - Ø 0.5% for answering oral questions posed by the tutors after completing the lab tasks.
- If a lab is missed the tasks may be completed out of the lab session and presented to the tutors at the start of the following lab.
- Work submitted later than the following lab session will not be marked without a formal special consideration application.

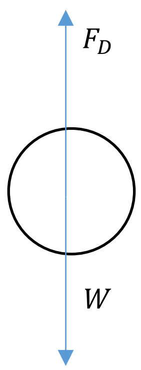
Lecture 1. Wednesday, 2 August 2017

Approximate Solution by Trial Functions

- Nice intro to FE method. Residual equation, shape function, solution error
- Useful in its own right
- Critical thinking for engineering problems

Example 1.1: mass falling through air

A ball of mass m falls through air from a great height. The ball has an initial velocity of $v = 0$ at time $t = 0$. From Newton's law we know the governing equation



$$m\frac{dv}{dt} = mg - bv^2$$

(where $b = \frac{1}{2}\rho A C_D$

- Exact solution is $v = \left(\frac{mg}{b}\right)^{\frac{1}{2}} \tanh\left[\left(\frac{bg}{m}\right)^{\frac{1}{2}} t\right]$

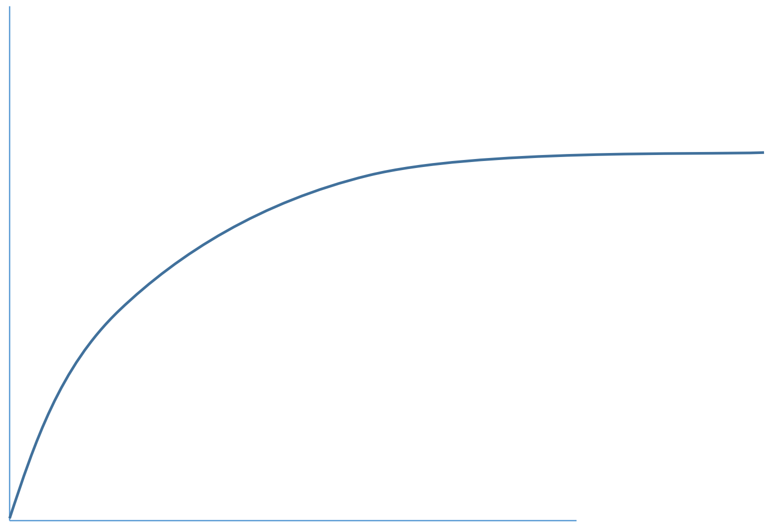
Approximate solution method:

- Trial function and collocation

1. Guess solution

Use intuition and consideration to guess solution shape.

- Graphing above, we know we:
    - Start at 0
    - Accelerate
    - approach v_∞
    - each dt, velocity will be decreasing a bit



We could think of some solutions which might fit:

- exponential $\tilde{v}(t) = v_\infty\left(1 - e^{-\frac{t}{\tau}}\right)$
- quadratic: $\tilde{v}(t) = c_1 + c_2 t + c_3 t^2$
- sinusoidal: $\tilde{v}(t) = c_1 \sin \omega t$
- higher order polynomial

2. choose trial funciton

using our intuition, we look at the exponential; where τ is a time constant which we want to find

$$\tilde{v} = v_\infty\left(1 - e^{-\frac{t}{\tau}}\right)$$

3. obtain residual equation

substitute $\tilde{v} \rightarrow v$ into the DE

$$m\frac{dv}{dt} = mg - bv^2$$

Becomes

$$m\frac{d\tilde{v}}{dt} \cong mg - b\tilde{v}^2$$

using our trial function $\tilde{v} = v_\infty\left(1 - e^{-\frac{t}{\tau}}\right) \rightarrow \frac{v_\infty}{\tau}e^{-\frac{t}{\tau}}$ gives

$$m\frac{v_\infty}{\tau}e^{-\frac{t}{\tau}} \cong mg - bv_\infty^2\left(1 - e^{-\frac{t}{\tau}}\right)^2$$

This is an approximation, to make it exact; we need to add the residual $\mathcal{R}$

$$m\frac{v_\infty}{\tau}e^{-\frac{t}{\tau}} = mg - bv_\infty^2\left(1 - e^{-\frac{t}{\tau}}\right)^2 + \mathcal{R}$$

To simplify the equation, we know that the terminal velocity is when $F = 0 \rightarrow mg = bv_\infty^2$; so we can sub that in and cancel out m

$$\frac{1}{\tau}\sqrt{\frac{mg}{b}}e^{-\frac{t}{\tau}} = g - g\left(1 - e^{-\frac{t}{\tau}}\right)^2 + \mathcal{R}$$

4. collocation

we want this to have a residual of 0 somewhere (in general, we can only do a finite number of points).

- The process of finding τ to make $\mathcal{R} = 0$ is **collocation**

For this problem; we'll try $\mathcal{R} = 0$ halfway between 0 and v_∞; so when $e^{-\frac{t}{\tau}} = 0.5$

- This is the collocation step

Substituting this into our DE:

$$\frac{1}{\tau}\sqrt{\frac{mg}{b}}(0.5) = g - g(1 - 0.5)^2 + 0$$

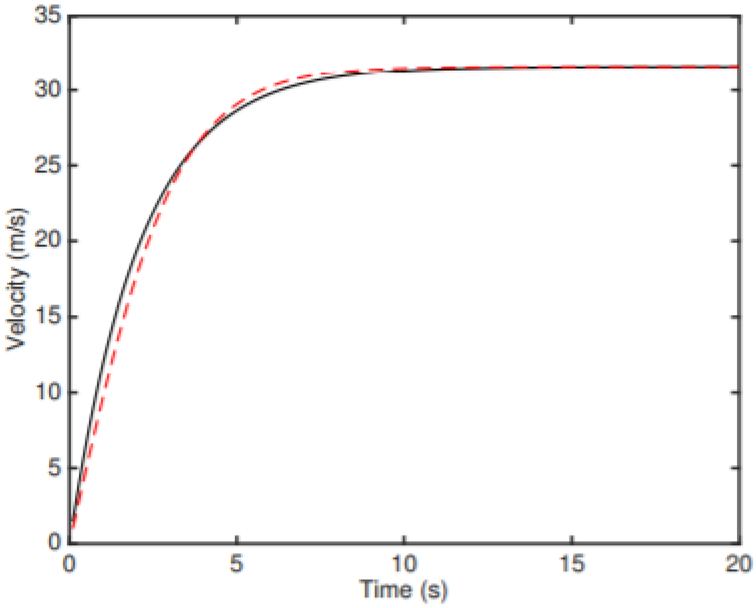
Solving this, we get that $\tau = 0.67\left(\frac{m}{gb}\right)^{\frac{1}{2}}$

So we can substitute this into our trial function to get

$$v \cong \tilde{v} = \sqrt{\frac{mg}{b}}\left(1 - e^{-\frac{t}{0.67\left(\frac{m}{gb}\right)^{\frac{1}{2}}}}\right)$$

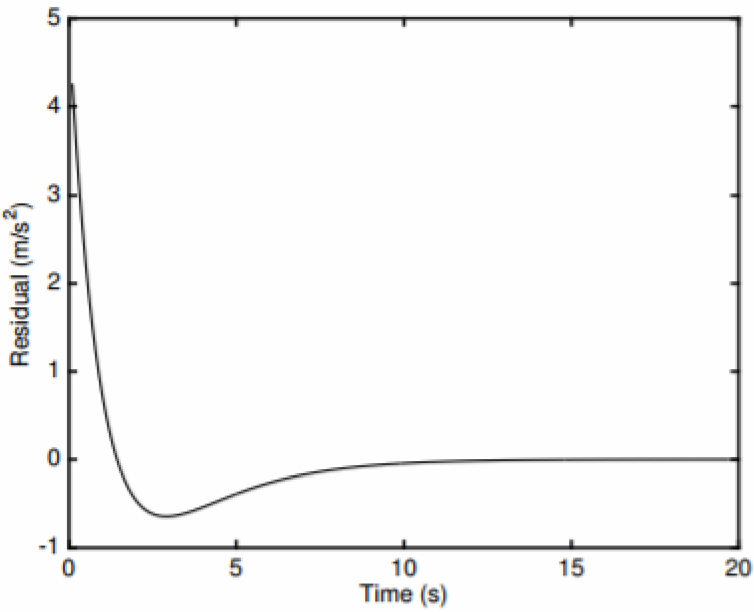
Comparing to exact solution:

- Approximate in solid; exact in dashed

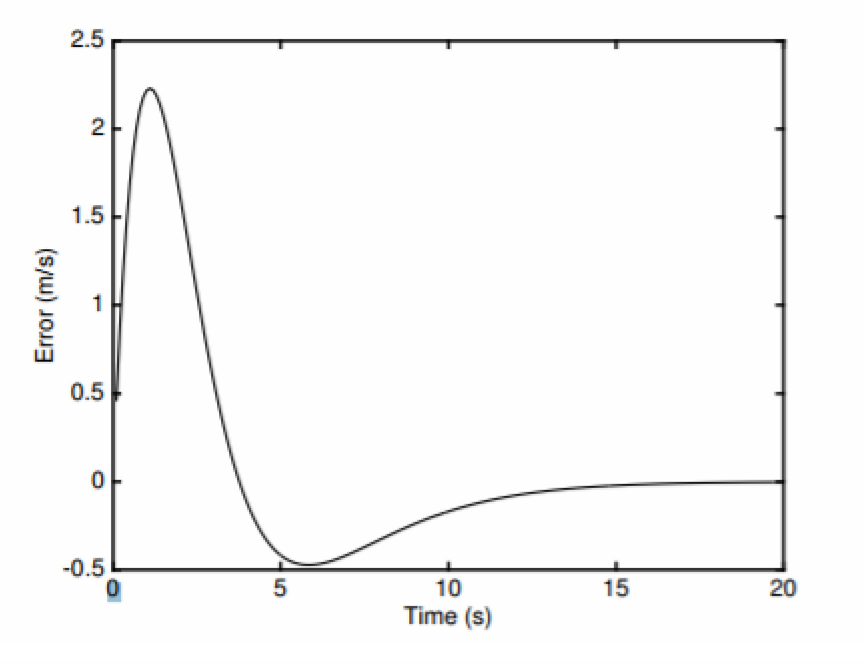


(a)

- We can see where we set the residual to be 0



- And error $\tilde{v} - v$



- Note that residual is error in the governing DE equation; whereas error is the difference in the final solution (2 different things)
  - In the trial function and collocation method; we try and minimise the residual (and not error)

L2 norm

We can compare different approximations by calculating the L_2 norm of the error:

$$\text{scaled } L_2 \text{ norm of solution error} = \frac{1}{J}\|E\|_2 = \frac{1}{J}\left(\sum_{j=1}^{J}\left(v(x_j) - \tilde{v}(x_j)\right)^2\right)^{\frac{1}{2}}$$

for J discrete locations along the rod.

Practice problem:

$$\frac{du}{dt} - \frac{1}{2} + u = 0$$

Domain is $u \in [0,1]$; with $u(t=0) = 1$. Try a second order trial function $\tilde{v}(t) = 1 - t + at^2$ (note that this solves the boundary condition)

$$\therefore \frac{d\tilde{v}}{dt} - \frac{1}{2} + \tilde{v} = R$$

$$(-1 + 2at) - \frac{1}{2} + (1 - t + at^2) = R$$

$$at^2 + 2at - t - \frac{1}{2} = R$$

Make collocation point $t = 0.5; R = 0$

$$a(0.5)^2 + 2a(0.5) - (0.5) - \frac{1}{2} = 0$$

Giving $a = 0.8$

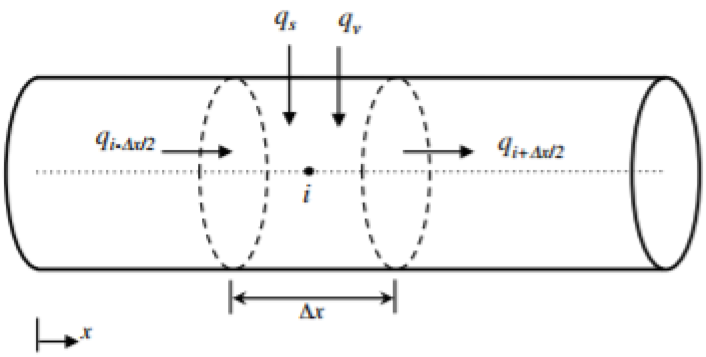
$$\therefore \tilde{v} = 1 - t + 0.8t^2$$

Lecture 2.

The heat equation

1D rod:

We will be looking at many heat transfer problems in this course. The governing equation for these systems is called the 'Heat Equation'. It can be derived using an energy balance on a control volume. Consider a round metal rod shown below.



Consider a control volume of Δx, surface area $A_s = 2\pi R\delta x$; cross sectional area $A_c = \pi R^2$ and volume $V = \Delta x A_c$.

The heat in is Q_{in} and out Q_{out}, over Δt time the temperature increases by ΔT

We have heat transfer of:

- Conduction in and out by $q_{i-\frac{\Delta x}{2}}$; $q_{i+\frac{\Delta x}{2}}$ (W/m^2)
- Through outer rod surface q_s (W/m^2) (radiation and convection)
- Internal source heating q_v (W/m^3) (eg, electrical current heating)

The heat balance can be written as:

$$Net\ heat = E_{in} - E_{out}$$
$$\rho C_p V \Delta T = \Delta t(Q_{in} - Q_{out})$$
$$[net\ gain = internal\ conduction + heating\ through\ surface + internal\ heat\ source)$$

- Density ρ

- C_p specific heat of rod material (J/kgK)

Giving:

$$\rho C_p V \Delta T = \Delta t \left(\left[q_{i-\frac{\Delta x}{2}} - q_{i+\frac{\Delta x}{2}} \right] A_c + q_s A_s + q_v V \right)$$

So that

$$\rho C_p A_c \frac{\Delta T}{\Delta t} = \left[-\frac{\Delta q_i}{\Delta x} \right] A_x + q_s 2\pi R + q_v A_c$$

So that, as $\Delta x \to \partial x; \Delta t \to \partial t$

$$\rho C_p A_c \frac{\partial T}{\partial t} = \left[-\frac{\partial q_i}{\partial x} \right] A_x + q_s 2\pi R + q_v A_c$$

Fourier's law of conduction says that

$$\boxed{q_i = -\kappa \frac{dT}{dx} \Big|_i}$$

So we substitute

$$\rho C_p A_c \frac{\partial T}{\partial t} = \left[-\frac{\partial}{\partial x} \left(-\kappa \frac{dT}{dx} \right) \right] A_x + q_s 2\pi R + q_v A_c$$

Simplifying into

$$\frac{\partial T}{\partial t} = \frac{\kappa}{\rho C_p} \frac{\partial^2 T}{\partial x^2} + \frac{q_s}{\rho C_p} \frac{2}{R} + \frac{q_v}{\rho C_p}$$

We let $\frac{\kappa}{\rho C_p} = \alpha$ (the thermal disfussivity) and get that:

$$\boxed{\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + Source\ terms}$$

Or in 3D:

$$\boxed{\frac{\partial T}{\partial t} = \alpha \nabla^2 \frac{\partial^2 T}{\partial x^2} + Source\ term}$$

Boundary Conditions

There are two common boundary conditions which we will encounter.

- Dirichlet
  - Fixed boundary condition
    - Eg: temperature defined at boundary $T(x = L) = 40°$
- Neumann

- Gradient at boundary is defined
  - Eg: constant heat conduction $q_s(x = L)$ is known, so that $\frac{dT}{dx}(x = L) = -\frac{q_s(x=L)}{\kappa}$
  - Eg: insulated boundary $q_x(x = L) = 0$, as heat conduction is $q_x = -\kappa \frac{dT}{dx}\Big|_x$; we get $\frac{dT}{dx}\Big|_{x=L} = 0$

Example 2.1:

A round metal rod of length L is held at T = 0◦C at x = 0 and T = 100◦C at x = 1 m. An electrical current is passed through the rod generating an internal heat source of 10 kW/m3 . With no heat transfer through the rod surface, the governing equation is,

$$\frac{\partial T}{\partial t} = \frac{\kappa}{\rho C_p}\frac{\partial^2 T}{\partial x^2} + \frac{q_v}{\rho C_p}$$

Assuming steady state, gives

$$\frac{\kappa}{\rho C_p}\frac{\partial^2 T}{\partial x^2} + \frac{q_v}{\rho C_p} = 0$$

Solution with second order trial function

$$T \cong \tilde{T}(t) = C_1 + C_2 x + C_3 x(x - L)$$

As $T(0) = 0; T_L = 100$; we get $C_1 = 0; C_2 = \frac{100}{L}$

$$\tilde{T}(t) = \frac{100}{L}x + C_3 x(x - L)$$

Giving the residual equation of:

$$R = \frac{\kappa}{\rho C_p}\frac{\partial^2 \tilde{T}}{\partial x^2} + \frac{q_v}{\rho C_p} = \frac{\kappa}{\rho C_p}(2C_3) + \frac{q_v}{\rho C_p}$$

- As this has no x dependence, we can set $R = 0 \forall x$; giving $C_3 = -\frac{q_v}{2\kappa}$
- This is the exact analytical solution

$$\tilde{T} = \frac{100}{L}x - \frac{q_v}{2\kappa}x(x - L)$$

This remarkable result demonstrates the power of this approach. The trial function method (and Finite Element method) is one of the few approximate methods where the exact result can be obtained. We would rarely, if ever expect to see this however. If we were to change this problem slightly, such as making qv a function of T, then this trial function would no longer be exact, but it might still be a very good approximation.

Example 2.2:

A round metal rod of length L = 1m is held at T = 100◦C at x = 0 and is insulated at x = 1m, so dT/dx = 0. The rod is cooled air flow from a fan with T_a = 0◦C with heat transfer coefficient h (W/m2K).

Under steady conditions, with $q_s = h(T - T_a)$, the unsteady heat equation becomes

$$0 = \kappa \frac{\partial^2 T}{\partial x^2} - \frac{2h(T - T_a)}{R}$$

Letting $\gamma^2 = \frac{2h}{\kappa R}$, we get

$$0 = \frac{\partial^2 T}{\partial x^2} - \gamma^2 T$$

Say for $\gamma^2 = 4$:

Parabolic trial function

$$\tilde{T} = 100 + C_3 x(x - 2)$$

Giving residual

$$\mathcal{R} = \frac{\partial^2 \tilde{T}}{\partial x^2} - \gamma^2 \tilde{T} = 2C_3 - \gamma^2\big(100 + C_3 x(x - 2)\big)$$

As $x \in [0,1]$; try $\mathcal{R} = 0 \;\; at$ x=0.5, giving

$$C_3 = \frac{400\gamma^2}{8 + 3\gamma^2}$$

Cubic trial function

$$\tilde{T} = 100 + C_3 x(x - 2) + C_4 x(x^2 - 3)$$

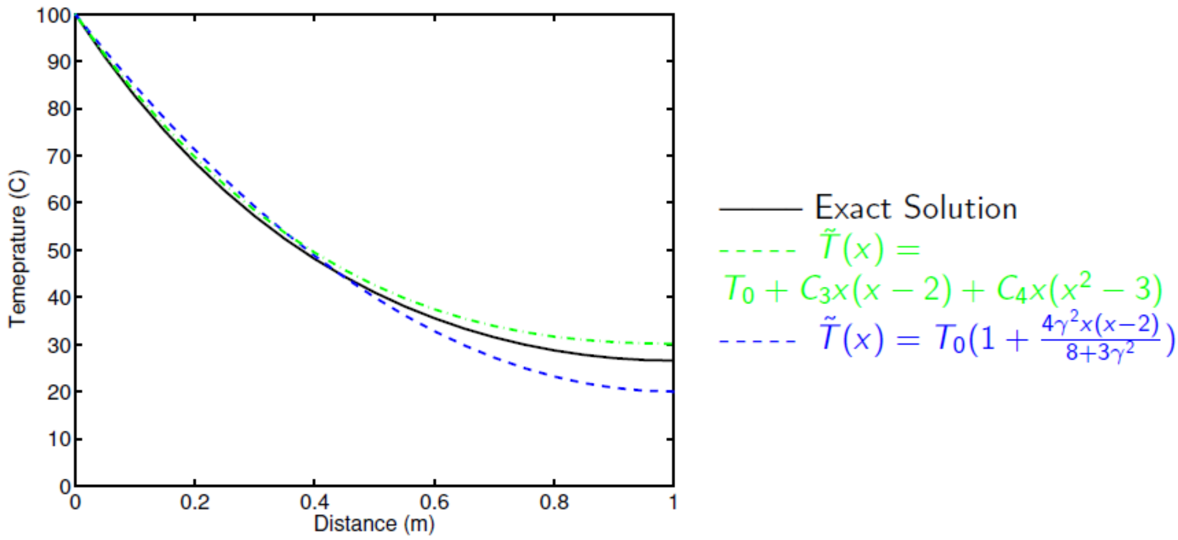
Again, giving the residual equation of

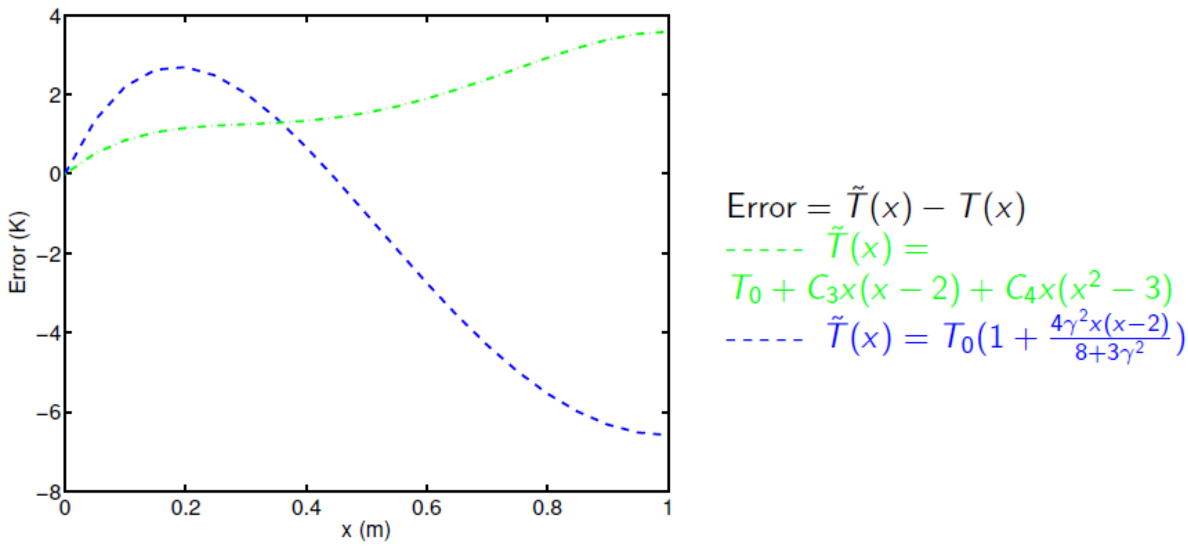
$$\mathcal{R} = \frac{\partial^2 \tilde{T}}{\partial x^2} - \gamma^2 \tilde{T} = 2C_3 + 6C_4 x - \gamma^2\big(100 + C_3 x(x - 2) + C_4 x(x^2 - 3)\big)$$

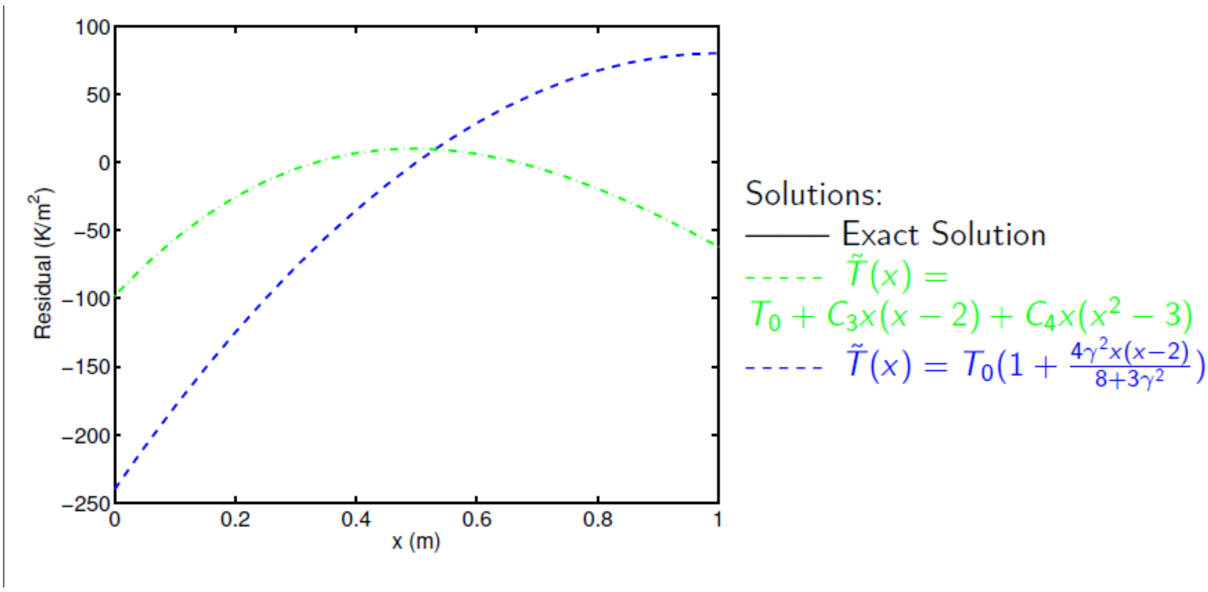
- As this has two unkowns, we need to collocate twice. Choose $x = \frac{1}{3}; \frac{2}{3}$ and solve to find

$$C_3 = 150.9; C_4 = -40.55$$

We can compare quadratic and cubic to exact solutions:


Temeprature (C)
Distance (m)
Exact Solution
$\tilde{T}(x) = T_0 + C_3x(x-2) + C_4x(x^2-3)$
$\tilde{T}(x) = T_0(1 + \frac{4\gamma^2 x(x-2)}{8+3\gamma^2})$


Error (K)
x (m)
Error = $\tilde{T}(x) - T(x)$
$\tilde{T}(x) = T_0 + C_3x(x-2) + C_4x(x^2-3)$
$\tilde{T}(x) = T_0(1 + \frac{4\gamma^2 x(x-2)}{8+3\gamma^2})$


Residual (K/m^2)
x (m)
Solutions:
Exact Solution
$\tilde{T}(x) = T_0 + C_3x(x-2) + C_4x(x^2-3)$
$\tilde{T}(x) = T_0(1 + \frac{4\gamma^2 x(x-2)}{8+3\gamma^2})$

Looking at the L_2 norm of each:

Parabolic Approximation	Cubic Approximation
L2 Norm of solution error = 0.82 K	L2 Norm of solution error = 0.46 K
L2 Norm of Residual = 23 K/m^2	L2 Norm of Residual = 8.3 K/m^2

So we see the cubic approximation is better, as it has the smaller norm

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Lecture 3. Wednesday, 9 August 2017

Method of Weighted Residuals

In the collocation methods before, we always set the residual $R = 0$ at some x value. What if we wanted to increase the accuracy of the method?

We could do this by

- Increasing the order of the polynomial; $\tilde{T} = c_1 + c_2 x + c_3 x^2 + c_4 c^3 + \cdots$
- Method of residual minimisation
    - Collocation
    - Sub-domain
    - Galerkin
- Number of approximations using piece wise solution

Residual minimisation

Say we have a trial function

$$\tilde{T} = C_1 x + C_2 x^2 + \cdots + C_n x^n$$

Collocation:

The residual is minimised at discrete points only

$$\mathcal{R}(x_i) = 0; \ \ \forall i = 1,2,..,n$$

Subdomain:

- The solution space is divided into 'subdomains', and the residual equally weighted in each domain

$$\int_{\Omega_i} \mathcal{R}(x) dx = 0; \ \ \forall i = 1,2,\ldots,n$$

Galerkin:

The residual is weighted by a function $N_i(x)$

$$\int_{\Omega} \mathcal{R}(x) N_i(x) dx = 0; \ \ \forall i = 1,2,\dots,n$$

Where $N_i(x)$ is a shape function which weights the residuals.

Shape function

In galerkin method, if $\tilde{T} = \sum_{i=1}^{n} C_i x^i$, then $N_i = x^i$

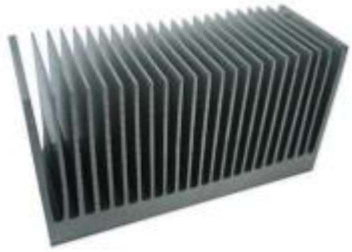
Eg:

$$\tilde{T}(x) = c_1 x + c_2 x^2 + c_3 x^3$$

Then

$$N_1 = x; N_2 = x^2; N_3 = x^3$$

Example 2.2 (again): by different methods



A round metal rod of length L = 1m is held at T = 100◦C at x = 0 and is insulated at x = 1m, so dT/dx = 0. The rod is cooled air flow from a fan with T_a = 0◦C with heat transfer coefficient h (W/m2K).

Under steady conditions, with $q_s = h(T - T_a)$, the unsteady heat equation becomes

$$0 = \kappa \frac{\partial^2 T}{\partial x^2} - \frac{2h(T - T_a)}{R}$$

$$\tilde{T} = 100 + C_3 x(x - 2L)$$

Letting $\gamma^2 = \frac{2h}{\kappa R}$, we get

$$0 = \frac{\partial^2 T}{\partial x^2} - \gamma^2 T$$

Say for $\gamma^2 = 4$:

We get the residual equation:

$$R = \frac{d^2 \tilde{T}}{dx^2} - \gamma^2 \, \tilde{T}$$

Collocation:

As $x \in [0,1]$; try $\mathcal{R} = 0 \ \ at$ x=0.5, giving

$$C_3 = \frac{400\gamma^2}{8 + 3\gamma^2}$$

Subdomain

$$\int_{\Omega_i} \mathcal{R}(x)dx = 0; \ \ \forall i = 1,2, \dots, n$$

As there is one unkown, only 1 equation is required (one domain), so the subdomain is all $x \in [0, L]$

$$\int_{x=0}^{L} R(x)dx = \int_{x=0}^{L} \frac{d^2\tilde{T}}{dx^2} - \gamma^2 \, \tilde{T}dx = 0$$

$$\therefore \int_{0}^{L} \left(2C - \gamma^2\big(100 + Cx(x - 2L)\big)\right) dx = 0$$

$$\rightarrow C = \frac{100\gamma^2}{2 + \frac{2}{3}\gamma^2 L^2}$$

- Different to the constant from collocation

Gelarkin:

For

$$\tilde{T} = 100 + C_3 \underbrace{\boxed{x(x - 2L)}}_{N_1}$$

There is only 1 unkown, and so $n = 1$; with the final term of

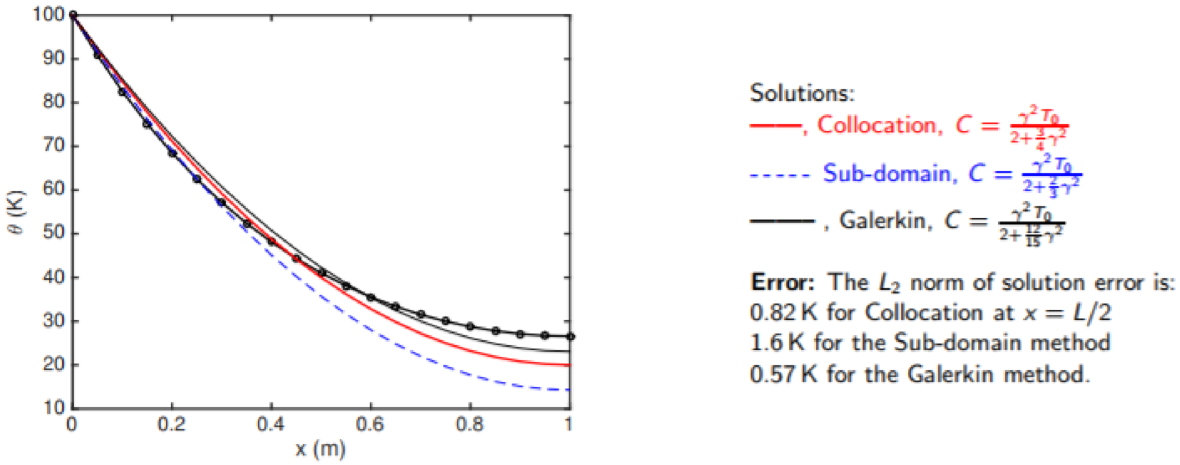
$$N_1 = x(x - 2L)$$

$$\therefore \int_{\Omega} R(x)N(x)dx = \int_{0}^{L} \left(2C - \gamma^2\big(100 + Cx(x - 2L)\big)\right)\big(x(x - 2L)\big)dx = 0$$

Integrating, and solving to get

$$C = \frac{100\gamma^2}{2 + \frac{4}{5}\gamma^2 L^2}$$

Solution Comparrison:



- L2 norm of the galerkin is smallest, followed by collocation and subdomain (which is very poor)

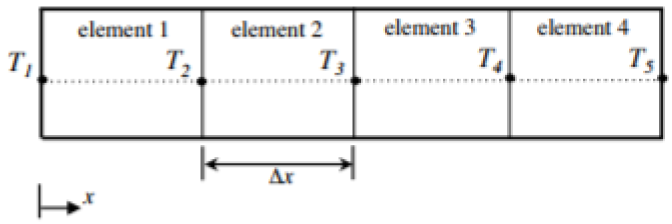
- Galerkin is a very good approach

Lecture 4. Thursday, 10 August 2017

Galerkin Finite Element Method

- Piece wise solution:

Use piece-wise solution e.g. Finite Element Method. Can improve the accuracy



Solution: define trial function

- Use a linear trial function.

Eg: for element 1, we can approximate

$$T(x) = c_1 x + c_2 x$$

Where $c_{1,2}$ are unkown.

But: a better way is:

- We known at $x_1; T(x_1) = T_1$, and so

$$T_1 = C_1 + C_2x_1$$
$$T_2 = C_1 + C_2x_2$$

Which we can eliminate $C_{1,2}$ to find

$$T(x) = \frac{x_2 - x}{x_2 - x_1}T_1 + \frac{x - x_1}{x_2 - x_1}T_2$$

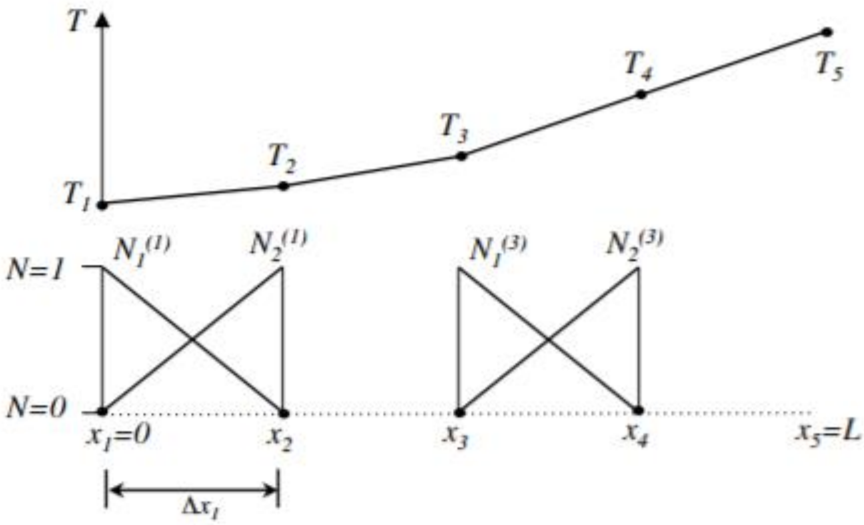
or

$$\boxed{T(x) = N_1(x)T_1 + N_2(x)T_2}$$

With

$$\boxed{N_1 = \frac{x_2 - x}{x_2 - x_1}; \text{and } N_2 = \frac{x - x_1}{x_2 - x_1}}$$

The global temperature is described by piece-wise linear functions for each element. Now we just have to obtain our residual equation and solve to find $T_{2,3,4}$



Weighted residuals

We use the Galerkin method to find the nodal values of T_i (our unkown fitting constants)

- For this example on element (1), we need 2 equations for T_1 and T_2

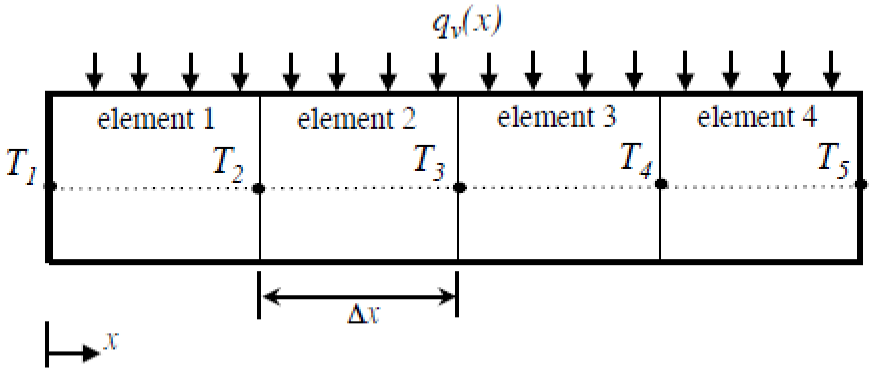
$$\int_{\Omega(1)} R(x)N_i(x)dx$$

Example 4.1:

Recall the heated rod problem from example 2.1. A round metal rod of length L is held at T = 0◦C at x = 0 and T = 100◦C at x = 1m. An electrical current is passed through the rod generating an internal heat source of qv = 100 kW/m3. If there is no heat transfer through the rod surface, the governing equation for the steady state solution can be written, where = qv/ = 2000 k/m2 with = 50W/mk. Find T(x) using the finite element Galerkin method using linear trial functions. Discretise the domain into four elements of equal length x.

solution:

rod divided into 4 elements:



- The temperateraures at $x = 0; L$ are given as $T_1 = 0; T_5 = 100$. We want to find $T_{2,3,4}$, and we can linearly interpolate between all T

Eg: If our residual equation is

$$R = \frac{d^2T}{dx^2} + \gamma$$

We have the substitution

$$\int_{\Omega(1)} \left(\frac{d^2T}{dx^2} + \gamma\right) N_i(x)dx = 0;\ \ i = 1,2$$

- In finite element method; we can only solve a DE of the same order of our trial function polynomial.
- To get around this, we'll need to use integration by parts.

For the first element: we define the linear trial function

$$T(x) = c_1 + c_2 x$$

Finding that:

$$T(x) = \frac{x_2 - x}{x_2 - x_1} T_1 + \frac{x - x_1}{x_2 - x_1} T_2$$

or

$$T(x) = N_1(x)T_1 + N_2(x)T_2$$

Where

$$N_1 = \frac{x_2 - x}{x_2 - x_1}; \text{and } N_2 = \frac{x - x_1}{x_2 - x_1}$$

- $N_{1,2}$ are the shape function or interpolation function of the element (or basis function)

$$x = x_1, N_1 = 1, N_2 = 0; x = x_2, N_1 = 0, N_2 = 1$$

$$N_1 + N_2 = 1$$

For this problem, every element has 2 interpolation functions, which we distinguish with $N_1^{(j)}$

For the other elements:

$$T^{(2)}(x) = N_1^{(2)}(x)T_2 + N_2^{(2)}(x)T_3$$
$$T^{(3)}(x) = N_1^{(3)}(x)T_3 + N_2^{(3)}(x)T_4$$
$$T^{(4)}(x) = N_1^{(4)}(x)T_4 + N_2^{(4)}(x)T_4$$

- A global temperature is now described by piecewise linear functions for each element

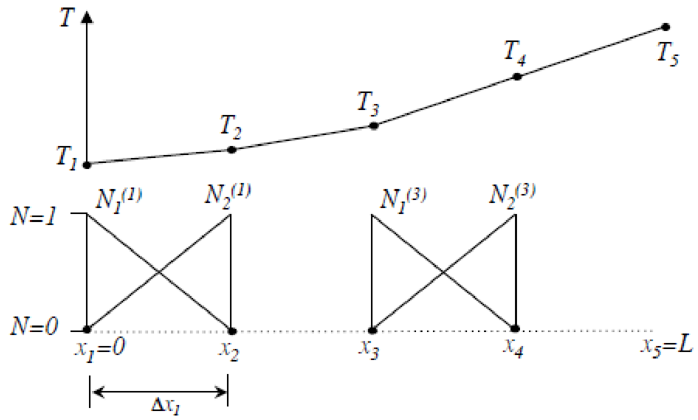


Figure 2: An example piece-wise linear solution for $T(x)$ is illustrated together with interpolation functions for elements (1) and (3).

Obtaining weighted residuals

We now use Galerkin to find the nodal values of T

Above: we have that

$$\int_{\Omega(1)} \left(\frac{d^2T}{dx^2} + \gamma\right) N_i(x)dx = 0; \;\; i = 1,2$$

So that:

$$\int_{x_1}^{x_2} \frac{d^2T}{dx^2} N_i(x)dx + \int_{x_1}^{x_2} \gamma N_i(x)dx = 0; i = 1,2$$

We can write $\int_{x_1}^{x_2} \frac{d^2T}{dx^2} N_i(x)dx = \left[N_i(x)\frac{dT}{dx}\right]_{x_1}^{x_2} + \int_{x_1}^{x_2} \gamma N_i(x)dx\,; i = 1,2$, so that overall

$$\int_{x_1}^{x_2} \frac{dT}{dx}\frac{dN_i(x)}{dx}dx = \left[N_i(x)\frac{dT}{dx}\right]_{x_1}^{x_2} + \int_{x_1}^{x_2} \gamma N_i(x)dx\,; \;\; i = 1,2$$

- $\left[N_i(x)\frac{dT}{dx}\right]_{x_1}^{x_2}$ is the boundar condition
- $\int_{x_1}^{x_2} \gamma N_i(x)dx$ is a forcing function

We need $\frac{dT}{dx}; \frac{dN_i}{dx}$ to solve this:

$$N_1 = \frac{x_2 - x}{x_2 - x_1}; \text{and } N_2 = \frac{x - x_1}{x_2 - x_1}$$

So we get

$$\frac{dN_1}{dx} = -\frac{1}{x_2 - x_1}; \frac{dN_2}{dx} = \frac{1}{x_2 - x_1}$$

As $T = N_1T_1 + N_2T_2$

$$\frac{dT}{dx} = \frac{dN_1}{dx}T_1 + \frac{dN_2}{dx}T_2 = -\frac{1}{x_2 - x_1}(T_1 - T_2)$$

- This indicates that the temperature gradient across each element is constant; as $T_1 - T_2$ is constant. (we assumed a linear function, so this is obvious)

Substituting this in, we get that: for $i = 1$

$$\int_{x_1}^{x_2} \frac{dT}{dx}\frac{dN_i(x)}{dx}dx = \left[N_i(x)\frac{dT}{dx}\right]_{x_1}^{x_2} + \int_{x_1}^{x_2} \gamma N_i(x)dx$$

$$\int_{x_1}^{x_2} -\frac{T_1 - T_2}{x_2 - x_1}\frac{-1}{x_2 - x_1}dx = N_1(x)\frac{dT}{dx}\mid_{x_2} - N_1(x)\frac{dT}{dx}\mid_{x_1} + \int_{x_1}^{x_2} \frac{\gamma(x_2 - x)}{x_2 - x_1}(x)dx$$

As $N_1(x_2) = 0; N_1(x_1) = 1$

$$\frac{1}{x_2 - x_1}(T_1 - T_2) = -\frac{dT}{dx}\mid_{x_1} + \gamma\frac{x_2 - x_1}{2}$$

Similarly, if $i = 2$

$$\frac{1}{x_2 - x_1}(-T_1 + T_2) = +\frac{dT}{dx}\mid_{x_2} + \gamma\frac{x_2 - x_1}{2}$$

- $\frac{dT}{dx}\mid_{x_1, x_2}$ (boundary gradients) are unkown; but they will largly disappear near the end

Matrix

We now want to do this for each element. To make this easy, we'll put this as a matrix:

$$\frac{1}{\Delta x}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} -\frac{dT}{dx}\mid_{x_1} \\ +\frac{dT}{dx}\mid_{x_2} \end{pmatrix} + \begin{pmatrix} \gamma\frac{\Delta x}{2} \\ \gamma\frac{\Delta x}{2} \end{pmatrix}$$