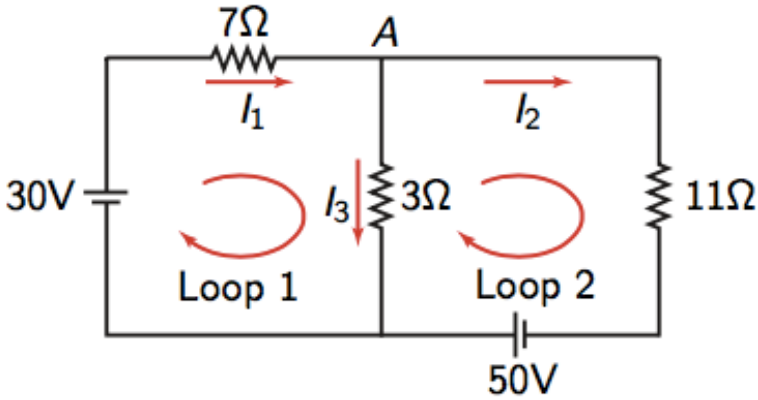
TOPIC 1 - LINEAR EQUATIONS

1.0 Linear Equations

1.01 Systems of Equations, Coefficient Arrays, Row Operations

One of the major topics studied in linear algebra is systems of linear equations and their solutions. We will study an efficient and systematic technique for solving simultaneous linear equations. We begin with an example to illustrate how linear equations can arise. We just set up the equations without solving them.

The numbers I_1, I_2, I_3 give the current (in Amps) as indicated in the diagram.



The current through point A gives:	$I_1 - I_2 - I_3 = 0$
The voltage drop around Loop 1 gives:	$7I_1 + 3I_3 = 30$
The voltage drop around Loop 2 gives:	$11I_2 - 3I_3 = 50$

Solving these three simultaneous equations gives the current flowing through all parts of the circuit. We've used Kirchoff's Laws and Ohm's Law in writing down the equations in this example, but you don't need to know what they are.

So in this example, the unknowns were I_1, I_2, I_3, and a typical equation obtained was $7I_1 + 3I_3 = 30$
We call this a **linear equation** in the variables I_1 and I_3 because the coefficients are constants and the variables are raised to the first power only. A linear equation in n variables x_1, x_2 and x_n is an equation of the form:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

Where a and b are constants, and not all the constants of a are zero. A finite collection of linear equations in the variables x_1, x_2 and x_n is called a system of linear equations or a **linear system**. Some examples of a linear system include:

$$\left\{\begin{array}{r} x + 2y = 7 \\ \frac{3}{8}x - 21y = 0 \end{array}\right\}$$

$$\left\{\begin{array}{r} x_1 + 5x_2 + 6x_3 = 100 \\ x_2 - x_3 = -1 \\ -x_1 + x_3 = 11 \end{array}\right\}$$

A solution to a system of linear equations in the variables x_1, x_2 and x_n is a set of values of these variables which satisfy every equation in the system. For the following linear system, there are a variety of ways to solve it.

$$\begin{Bmatrix} 2x - y = 3 \\ x + y = 0 \end{Bmatrix}$$

To solve graphically, one would need an accurate sketch and is not practical for three or more variables. It can be solved using elimination which will always give a solution, but is too adhoc, particularly in higher dimensions (meaning three or more variables).

The third way to solve this linear system is by coefficient arrays. As an example, consider the previously encountered linear system:

$$\begin{Bmatrix} I_1 & - & I_2 & - & I_3 & = 0 \\ 7I_1 & + & (0\times)I_2 & + & 3I_3 & = 30 \\ (0\times)I_1 & + & 11I_2 & - & 3I_3 & = 50 \end{Bmatrix}$$

The coefficients of the unknowns can be written as 3 x 3 array of numbers (known as a matrix), shown below.

$$\begin{bmatrix} 1 & -1 & -1 \\ 7 & 0 & 3 \\ 0 & 11 & -3 \end{bmatrix}$$

A **matrix** is a rectangular array of numbers. The numbers in the array are called the **entries** of the matrix. A p x q matrix has p rows and q columns. The **augmented matrix** for a linear system is the matrix formed from the coefficients in the equations and the constant terms.

Example 1

Write the following system of linear equations as an augmented matrix.

$$\begin{Bmatrix} 2x - y = 3 \\ x + y = 0 \end{Bmatrix}$$

As an augmented matrix:

$$A = \left[\begin{array}{cc|c} 2 & -1 & 3 \\ 1 & 1 & 0 \end{array}\right]$$

Note that the number of rows is equal to the number of equations. Each column, except the last, corresponds to a variable. The last column contains the constant term from each equation.

Our aim is to use matrices to assist us in finding a solution to a system of equations. First, we need to decide what sort of operations we can perform on the augmented matrix. An essential condition is that whichever operations we perform, we must be able to recover the solution to the original system from the new matrix we obtain.

The elementary **row operations** are interchanging two rows, multiplying a row by a non-zero constant and adding a multiple of one row to another.

When these elementary row operations are applied, the matrices may not be equal, but are equivalent in that the solution set is the same for each system represented by each augmented matrix.

1.02 Reduction of Systems to Reduced Row-Echelon Form

Using a sequence of elementary row operations, we can always get to a matrix that allows us to determine the solution set of a linear system. However, the degree of complication increases as the number of variables and equations increases, so it is a good idea to formalise the process. The leftmost non-zero element in each row is called the **leading entry**.

A matrix is in **row-echelon form** if:

1. For any row with a leading entry, all elements below that entry and in the same column as it, are zero.
2. For any two rows, the leading entry of the lower row is further to the right than the leading entry in the higher row.
3. Any row that consists solely of zeros is lower than any row with non-zero entries.

Some examples of matrices in row-echelon form includes:

$$\begin{bmatrix} 1 & -2 & 3 & 4 & 5 \end{bmatrix}, \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

An example of a matrix that is not in row-echelon form includes:

$$\begin{bmatrix} 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 3 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & -3 & 6 & -4 & 9 \end{bmatrix}$$

Gaussian elimination is a systematic (or algorithmic) approach to the reduction of a matrix to row-echelon form. This is achieved by following the steps below:

1. Make the top left element (row 1, column 1) a leading entry; that is, reorder rows so that the entry in the top left position is non-zero.
2. Add multiples of the first row to the other rows to make all other entries (from row 2 down) in the first column zero.
3. Reorder rows 2… n so that the next leading entry is in row 2.
4. Add multiples of the second row to rows 3…n, making all other entries (from row 3 down) in the column containing the second leading entry zero.
5. Repeat until you run out of rows.

The matrix should now be in row-echelon form.

Example 2

Use Gaussian elimination to reduce the augmented matrix which represents the linear system below to row-echelon form.

$$\left\{\begin{array}{r} 3x+2y-\ \ z=-15 \\ x+\ \ y-4z=-30 \\ 3x+\ \ y+3z=\ \ \ 11 \\ 3x+3y-5z=-41 \end{array}\right\}$$

$$=\begin{bmatrix} 1 & 1 & -4 & -30 \\ 3 & 2 & -1 & -15 \\ 3 & 1 & 3 & 11 \\ 3 & 3 & -5 & -14 \end{bmatrix}$$

$$=\begin{array}{r} \\ R_2-3R_1 \\ R_3-3R_1 \\ R_4-3R_1 \end{array}\begin{bmatrix} 1 & 1 & -4 & -30 \\ 0 & -1 & 11 & 75 \\ 0 & -2 & 15 & 101 \\ 0 & 0 & 7 & 49 \end{bmatrix}$$

$$=\begin{array}{r} \\ -R_2 \\ \\ \\ \end{array}\begin{bmatrix} 1 & 1 & -4 & -30 \\ 0 & 1 & -11 & -75 \\ 0 & -2 & 15 & 101 \\ 0 & 0 & 7 & 49 \end{bmatrix}$$

$$=\begin{array}{r} \\ \\ R_3+2R_2 \\ \\ \end{array}\begin{bmatrix} 1 & 1 & -4 & -30 \\ 0 & 1 & -11 & -75 \\ 0 & 0 & -7 & -49 \\ 0 & 0 & 7 & 49 \end{bmatrix}$$

$$=\begin{array}{r} \\ \\ R_3\div -7 \\ \\ \end{array}\begin{bmatrix} 1 & 1 & -4 & -30 \\ 0 & 1 & -11 & -75 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 7 & 49 \end{bmatrix}$$

$$=\begin{array}{r} \\ \\ \\ R_4-7R_3 \end{array}\begin{bmatrix} 1 & 1 & -4 & -30 \\ 0 & 1 & -11 & -75 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By reducing a matrix to row-echelon form, Gaussian elimination allows us to easily solve a system of linear equations. To do this, we need to read off the final equations from the row-echelon matrix and then manipulate them to find the solution. The final manipulation is sometimes called **back substitution**. We illustrate this in the example below.