TOPIC 7 - FUNCTIONS OF TWO VARIABLES

7.0 Functions of Two Variables

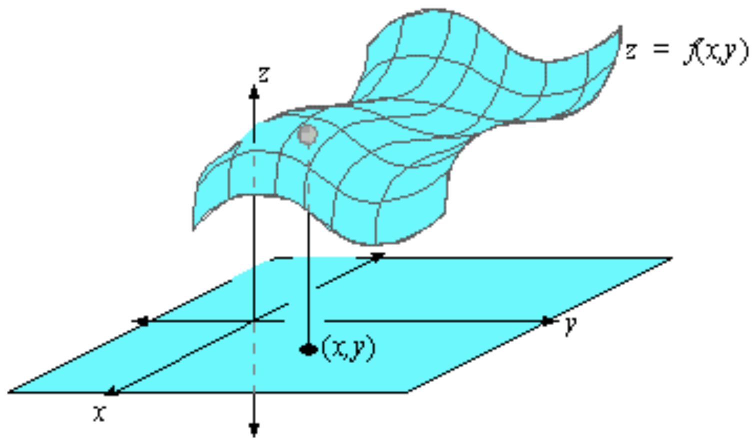
7.01 Sketching Functions of Two Variables

An example of a function of two variables is the temperature T at a point on the Earth's surface at a given time depends on the latitude x and the longitude y. We think of T being a function of the variables x, y (independent variables) and write $T = f(x, y)$. In general, a function of two variables is a mapping f that assigns a unique real number $z = f(x, y)$ to each pair of real numbers (x, y) in some subset D of the xy plane, where D is the **domain** of f.

We can represent the function f by its graph. The **graph** of f is:

$$\{(x, y, z) : (x, y) \in D \; and \; z = f(x, y)\}$$

This is a surface lying directly above the domain D. The x and y axes lie in the horizontal plane and the z axis is vertical.



The Cartesian form of a **plane** is:

$$ax + by + cz = d$$

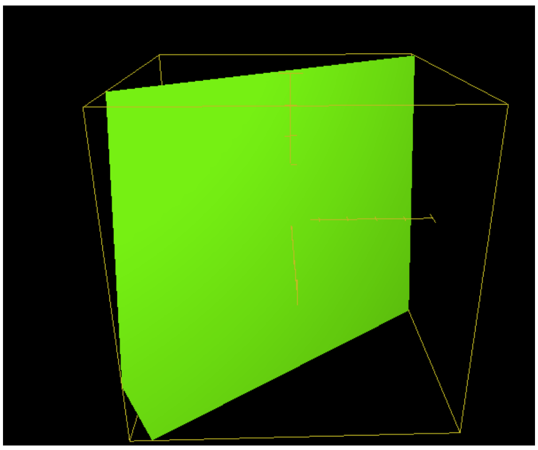
Where a, b, c and d are real constants. A **normal vector** to the plane is $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

Example 1

The plane $4x + 3y + z = 2$ can be written as $z = 2 - 4x - 3y$, so is the graph of the function given by $f(x, y) = 2 - 4x - 3y$. Sketch the plane.

Find the intercepts:

$$x - \text{int} \Rightarrow y = z = 0, so\ 4x = 2, x = 0.5\ and\ is\ the\ point\ (1/2, 0, 0)$$
$$y - \text{int} \Rightarrow x = z = 0, so\ 3y = 2, y = 2/3\ and\ is\ the\ point\ (0, 2/3, 0)$$
$$z - \text{int} \Rightarrow x = y = 0, so\ z = 2\ and\ is\ the\ point\ (0, 0, 2)$$



Note that the plane extends perpendicular to each face.

A curve on the surface $z = f(x,y)$ for which z is a constant is a **contour**. The same curve drawn in the xy plane is a level curve. Thus a **level curve** of f has the form:

$$\{(x,y){:}f(x,y) = c\}$$

The key steps in a drawing a graph of a function of two variables are:

1. Draw the x, y and z axes. For right handed axes: the positive x axis is towards you, the positive y-axis points to the right and the positive z axis points upward.
2. Draw the y-z cross section.
3. Draw some level curves.
4. Draw the x-z cross section.
5. Label any x, y and z intercepts and key points.

7.02 First Order and Second Order Partial Derivatives

We say that f has the limit L as (x, y) approaches a point:

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

If when (x, y) approaches the point along any path in the domain, then $f(x, y)$ gets arbitrarily close to L. Note that:

1. L must be finite.
2. The limit can exist if f is undefined at the point.
3. The usual limit laws apply.

Let f be a real-valued functions. Then f is continuous at (x, y) if:

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$$

Note that the continuity theorems for two functions of one variable can be generalised to functions of two variables.

Example 1

$$\textit{Let } f(x,y) = x^2 + y^2. \textit{ For which values of } x \textit{ and } y \textit{ is } f \textit{ continuous?}$$

$$\textit{Polynomials for } f(x,y) \textit{ have terms of form } ax^n y^m.$$

$$\textit{Since } f(x,y) \textit{ is a polynomial, it is continuous for all } (x,y) \in \mathbb{R}$$

Example 2

$$\textit{Evaluate } \lim_{(x,y)\to(2,1)} \log\left(1 + 2x^2 + 3y^2\right)$$

$$\begin{aligned}\lim_{(x,y)\to(2,1)} \log\left(1 + 2x^2 + 3y^2\right) &= \log\left(\lim_{(x,y)\to(2,1)} \left(1 + 2x^2 + 3y^2\right)\right) \\ &= \log\left(1 + 8 + 3\right) \\ &= \log\left(12\right)\end{aligned}$$

The **first order partial derivatives** of f with respect to the variables x and y are defined by the limits:

$$f_x = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

The partial derivative with respect to x measures the rate of change of f with respect to x when y is held constant. The partial derivative with respect to y measures the rate of change of f with respect to y when x is held constant.

Example 3

$$Let\ f(x,y) = 3x^3y^2 + 3xy^4.\ Find\ f_x\ and\ f_y$$

$$f_x = 9x^2y^2 + 3y^4$$

$$f_y = 6x^3y + 12xy^3$$

Example 4

$$Let\ f(x,y) = y\log x + x\tanh 3y.\ Find\ f_x\ and\ f_y\ at\ (1,0)$$

$$f_x = \frac{y}{x} + \tanh 3y$$

$$f_y = \log x + 3x\operatorname{sec}h^2 3y$$

$$So\ f_x(1,0) = 0 + \tanh(0) = 0$$

$$f_y(1,0) = \log 1 + 3\operatorname{sec}h^2 0 = 0 + 3 = 3$$

The second order partial derivatives of f with respect to x and y are defined by:

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Note that if the second order partial derivatives of *f* exist and are continuous then:

$$f_{xy} = f_{yx}$$

Example 5

Find the second order partial derivatives of:

$$f(x,y) = x^2 + 2x^3y^2 - 3y^4$$

$$f_x = 2x + 6x^2y^2$$

$$f_y = 4x^3y - 12y^3$$

$$f_{xx} = 2 + 12xy^2$$

$$f_{yy} = 4x^3 - 36y^2$$

$$f_{xy} = 12x^2y$$

$$f_{yx} = 12x^2y$$

$$f_{xy} = f_{yx} \text{ so } f \text{ is continuous.}$$

Example 6

Find the second order partial derivatives of the following function *f*.

$$f(x,y) = x(\sin x + 2y)$$

$$f_x = \sin(x+2y) + x\cos(x+2y)$$

$$f_y = 2x\cos(x+2y)$$

$$f_{xx} = 2\cos(x+2y) - x\sin(x+2y)$$

$$f_{yy} = -4x\sin(x+2y)$$

$$f_{xy} = 2\cos(x+2y) - 2x\sin(x+2y)$$

$$f_{yx} = 2\cos(x+2y) - 2x\sin(x+2y)$$

Note that the last two second order partial derivatives are equal as expected since trigonometric functions and polynomials are continuous for all (x, y) in their domain.

7.03 Tangent Planes and Differentiability

Let *f* be a real-valued function. We say that *f* is differentiable at a point if the tangent lines to all curves on the surface $z = f(x,y)$ pass through a point form a plane, called the **tangent plane**. This holds if the first order partial derivatives exist and are continuous near the point. The tangent plane has equation:

$$z - z_0 = f_x \big|_{(x_0,y_0)} (x - x_0) + f_y \big|_{(x_0,y_0)} (y - y_0)$$