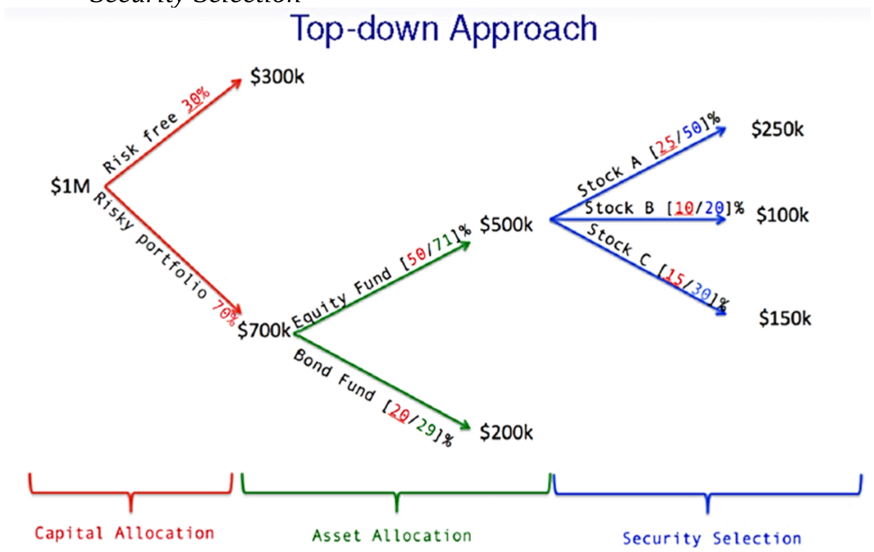
LECTURE 1: CAPITAL ALLOCATION

1.1 Introduction

Top-down Approach

- *Capital Allocation*
- *Asset Allocation*
- *Security Selection*



Lec1. How much risk?
Capital Allocation b/w the risky portfolio and the risk-free asset

Lec2. Which risk?
Asset Allocation across asset classes, e.g.
- US stocks
- International stocks
- Long term bonds

Lec3. Which risk?
Security Selection of individual assets within each asset class

Two-step Approach

1. Determine the *set* of feasible portfolios
   - The combinations of risk and expected-return available
2. Decide which one is the most preferred
   - The *most preferred* portfolio => the *highest utility*
     - Investor's goal
       - Save for retirement (30-40 years from now)
       - Save for house (10-15 years from now)
       - Become rich
     - Investor's preferences
       - Risk aversion
     - Investor's constraints
       - Liquidity constraints

Mean Variance Utility Function

Mean-Variance Utility Function is a way to assign score to competing portfolios on the basis of expected return, risk and risk aversion.

$$U = E(r) - \frac{1}{2}A\sigma^2$$

$E(r)$ expected return
A risk aversion
σ^2 risk

> Fair game: $E(r) = 0$ but $\sigma^2 \neq 0$, Sharpe ratio = 0, the one with an expected value equal to zero or costs its expected value for the right to play.
>
> If A>0, will avoid fair game, they accept risk only if risk premium exists.

- Utility is enhanced by high expected return $E(r)$
- The effect of risk σ^2 depends on risk attitude A
  - For a *risk averse* invevstor (A>0), risk decrease marginal utility of money, risk averser *rejects fair game*
  - For a *risk neutral* investor (A=0), risk does not affect utility (U=E(r), only care about expected return)
  - For a *risk seeking* investor (A<0), risk increases marginal utility of money
  - If A≠0, the larger the magnitude the larger the effect
- A mean variance investor *only* cares about risk and return, does not take into account skewness or kurtosis.
- Risk is different from *uncertainty*.

1.2 Expected Return & Risk

Interested in *future & unknown* returns; Not what *you* expect, it's what *market* expect

1. **Guess** Investors should care about real expected returns rather than nominal returns (due to inflation)
2. **Scenario Analysis (Educated Guess)**

Expected Return using Scenario Analysis

- Defintion

$$E(r) = \sum_{s=1}^{n}(\text{Probability of Scenario}) \times (\text{Possible Return}) = \sum_{s=1}^{n} p_s \times r_s$$

- Coles

Economy	Probability	Return
Good	35%	15%
Normal	50%	5%
Bad	15%	-10%

$$E(r) = 0.35 * 0.15 + 0.50 * 0.05 + 0.15 * (-0.10) = 0.0625$$

- Qantas

Economy	Probability	Return
Good	35%	50%
Normal	50%	7.5%
Bad	15%	-100%

$$E(r) = 0.35 * 0.5 + 0.5 * 0.075 + 0.15 * (-1) = 0.0625$$

Risk using Scenario Analysis

- Defintion

$$Var(r) = \sum_{s=1}^{n}(\text{Probability of Scenario}) \times (\text{Possible Return} - \text{ExpectedReturn})^2$$

$$= \sum_{s=1}^{n} p_s \times \left(r_s - E(r)\right)^2 = E[(r_s - E(r))^2]$$

- Coles

Economy	Probability	Return
Good	35%	15%
Normal	50%	5%
Bad	15%	-10%

$$Var(r) = .35 * (.15 - .0625)^2 + .50 * (.05 - .0625)^2 + .15 * (-.10 - .0625)^2 = 0.0067$$

- Qantas

Economy	Probability	Return
Good	35%	50%
Normal	50%	7.5%
Bad	15%	-100%

$$Var(r) = .35 * (.50 - .0625)^2 + .5 * (.075 - .0625)^2 + .15 * (-1 - .0625)^2 = 0.23$$

- *Why do we take squared derivatives? Why σ^2 not $se(r)$?*
  - the squares will remove the negative values giving a positive finite average
  - the square emphasizes larger differences (emphasizes extremes)
- **Shape Ratio (Reward-to-Volatility Ratio)**

  The Shape Ratio measures the *excess return* (or risk premium) *per unit of risk*.

  It measures how well the return of an asset *compensates* the investor for the risk taken.

  $Sh = \dfrac{E(r) - r_f}{\sigma_r}$ => the slope of CAL: The higher the Sharpe ratio the *steeper* the CAL

3. **Past data**

- Use the *sample mean* to estimate the *expected return*

$$E(r) = \frac{1}{N}\sum_{i=1}^{N} r_i = \bar{\mu} \qquad E(r) = \sum_{i=1}^{N} p_i \times r_i = \sum_{i=1}^{N} \frac{1}{N} r_i = \frac{1}{N}\sum_{i=1}^{N} r_i = \bar{\mu}$$

NB: Can be linked to scenario analysis, each observation as an equally likely scenario ($\bar{\mu}$ is the average mean)

- Use the *sample variance* to estimate how much *risk* to expect

$$V(r) = \frac{1}{N}\sum_{i=1}^{N}(r_i - \bar{\mu})^2$$

[VAR.P]: assume the whole population

- *Estimation error* – sample variance is *biased* downward as the deviations use sample average $\bar{\mu}$ instead of the unknown & true expected value $E(r)$ => Bias can be *eliminated* by multiplying by $N/(N-1)$

$$V(r) = \left(\frac{N}{N-1}\right)\frac{1}{N}\sum_{i=1}^{N}(r_i - \bar{\mu})^2 = \frac{1}{N-1}\sum_{i=1}^{N}(r_i - \bar{\mu})^2$$

[VAR.S]: a sample of population

As N increases, the difference b/w VAR.P and VAR.S approaches zero.

4. **CAPM and factor models** (Lecture 4 & 5)

- **Reality check**
  - *Risk is different from uncertainty* (risk can be expected, but real uncertainty cannot)
    - in the real world, no one uses scenario analysis => casinos probabilities are known & fix; real world probabilities are unknown & time-varying
  - *Which window do we use to compute the mean?*
    - Longest series available, we want to incorporate as much info as possible
    - Recent data, markets now are probably different from past
    - Or maybe use the full series but higher weight to the most recent observations
  - *How do we treat outliers?*
    - Bias out estimation? Might happen again? Outliers are the observations above the 99.99th, 99th, or 95th percentile?
  - *Dose the past repeat itself?* (Some maybe yes, others not)
  - *Borrow rate ≠ lend rate?* so CAL is not linear.

1.3 Risk Aversion

Quantify risk-aversion?

- ⇨ The guaranteed amount of money that an individual would view as equally desirable as a risky asset
- ⇨ The more you are willing to 'give up' in order not to face the risk, the more risk averse you are, the more willing to avoid 'fair game'.

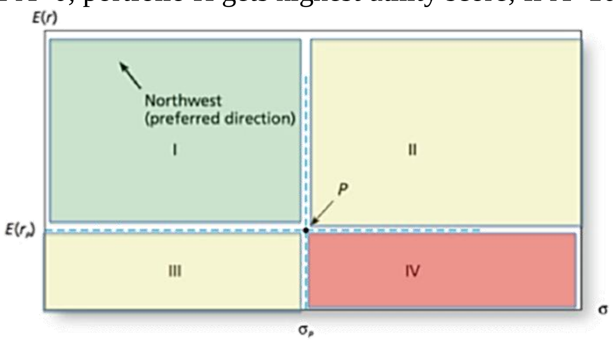
1.4 Mean Variance Utility Function

Utility of expected value ≠ Expected Utility
e.g. U(E[coin]) > E[U(coin)]
U(1M$) > 0.5*U(0)+0.5U(2M$)

Portfolio	Risk Premium	Expected Return	Risk (SD)
L (low risk)	2%	7%	5%
M (medium risk)	4	9	10
H (high risk)	8	13	20

Investor Risk Aversion (*A*)	Utility Score of Portfolio *L* [$E(r) = .07; \sigma = .05$]	Utility Score of Portfolio *M* [$E(r) = .09; \sigma = .10$]	Utility Score of Portfolio *H* [$E(r) = .13; \sigma = .20$]
2.0	$.07 - ½ \times 2 \times .05^2 = .0675$	$.09 - ½ \times 2 \times .1^2 = .0800$	**$.13 - ½ \times 2 \times .2^2 = .09$**
3.5	$.07 - ½ \times 3.5 \times .05^2 = .0656$	**$.09 - ½ \times 3.5 \times .1^2 = .0725$**	$.13 - ½ \times 3.5 \times .2^2 = .06$
5.0	$.07 - ½ \times 5 \times .05^2 = .0638$	**$.09 - ½ \times 5 \times .1^2 = .0650$**	$.13 - ½ \times 5 \times .2^2 = .03$

If A=0, portfolio H gets highest utility score; If A=10, portfolio L gets highest utility score



If A > 0, Utility increases going *North-West* (either standard deviation decreases or expected return increase or both)

If A < 0, Utility increases going *North-East*

If A = 0, Utility increases going *North*

Quadrant I, contains portfolios which are *better* than P because have higher expected return and lower risk

Portfolio	Standard Dev	Expected Return	A=2	A=4	A=6
P	10.0%	6.0%	5.0%	4.0%	3.5%
Q	10.0%	8.0%	7.0%	6.0%	5.5%
R	5.0%	6.0%	5.8%	5.5%	5.4%
S	5.0%	8.0%	7.8%	7.5%	7.4%

Quadrant IV, contains portfolios which are *worse* than P because have lower expected return and higher risk

Portfolio	Standard Dev	Expected Return	A=2	A=4	A=6
P	10.0%	6.0%	5.0%	4.0%	3.5%
Q	15.0%	6.0%	3.8%	1.5%	0.4%
R	10.0%	3.0%	2.0%	1.0%	0.5%
S	15.0%	3.0%	0.8%	-1.5%	-2.6%

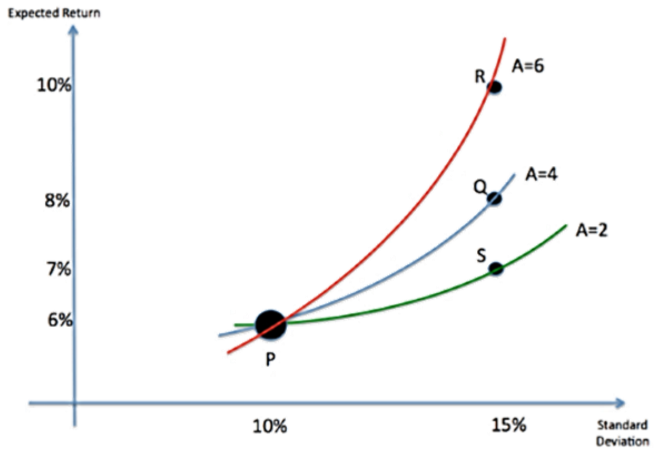
Desirability of portfolios in Quadrant II & III, compared to P, depends on *magnitude of risk aversion*

Quadrant III, contains portfolios which have lower expected return but also lower risk

Quadrant II, contains portfolios which have higher expected return but also higher risk

- Starting at P, an increase in SD lowers utility, it must be compensated for by an increase in expected return
- It depends on *risk aversion* => the higher A, the higher must be the expected return to have the same utility

Portfolio	Standard Dev	Expected Return	Utility A=2	Utility A=4	Utility A=6
P	10.0%	6.0%	5%	4%	3%
Q	15.0%	8.0%	6%	4%	1%
R	15.0%	10.00%	8%	6%	3%
S	15.0%	7.00%	5%	3%	0%



Indifferent curve:

Same utility for different $E[r]$ & σ^2

Slope > 0, ∵ $\sigma^2 \uparrow, E[r] \uparrow$

Slope ↑, ∵ marginal utility declines

PR indifferent line: same utility

(point P, R 对于 A=6 的投资者具有相同的吸引力)

when $\sigma^2 \uparrow$, to hold utility be the same, $E[r]$ has to ↑

$$U = E[r] - \frac{1}{2}A\sigma^2$$

1.5 Feasible Portfolios

Portfolios of one risky asset and a risk-free asset

$r_c = \omega r_p + (1-\omega) r_f$

- ω the fraction of wealth invested in the risky asset
- $1-\omega$ the remaining proportion invested in the risk-free asset

Expected return:

$E(r_c) = \omega E(r_p) + (1-\omega) r_f$

$E(r_c) = r_f + \omega[E(r_p) - r_f], Sharp = \dfrac{E[r_p] - r_f}{\sigma_p}$

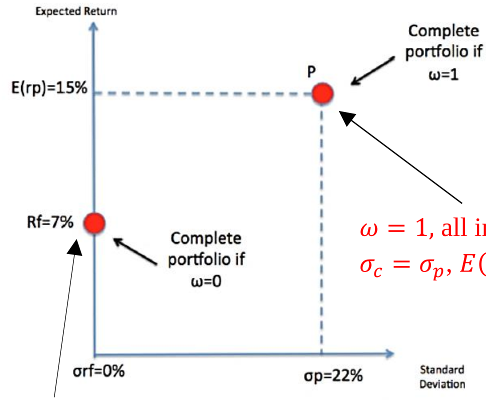
Variance: $\sigma_c^2 = \omega^2 \sigma_p^2 \;=>\; \omega = \sigma_c / \sigma_p$

- $var\,(kx) = k^2 var\,(x)$
- $\sigma_f = 0$

Treasury bills are commonly viewed as risk-free assets.
∵ SR: makes their values insensitive to i%, and inflation uncertainty over their time to maturity is negligible.

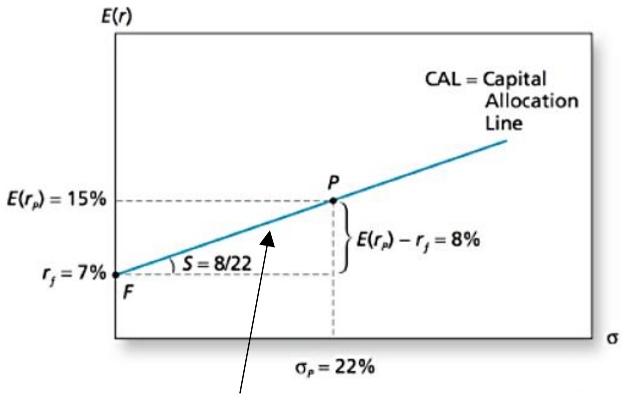
CAL: Capital Allocation Line

Investment opportunity set (the set of feasible portfolio resulting from different ω) => *Capital Allocation Line*



$\omega = 1$, all invest in risky asset
$\sigma_c = \sigma_p, E(r_c) = E(r_p)$

$\omega = 0$, all invest in risk-free asset
$\sigma_c = \sigma_f = 0$, intercept point $E(r_c) = r_f$



The midrange, ω lies b/w 0 and 1, graph on the straight line connecting the risk-free and the risky portfolio

CML

$E(r_c) = r_f + \omega[E(r_p) - r_f]$ (1)

$\omega = \dfrac{\sigma_c}{\sigma_p}$ (2)

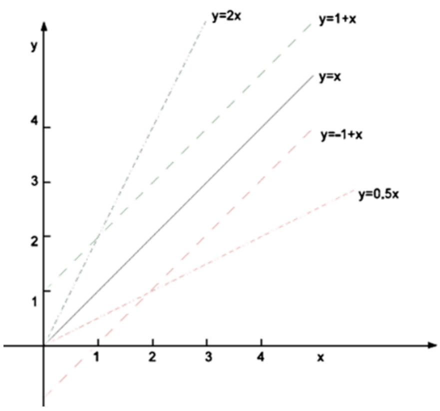
Substituting (2) in (1) we find:

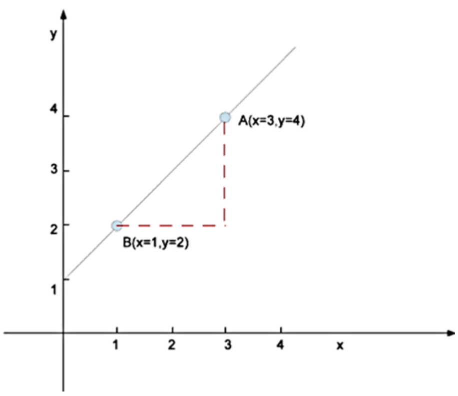
$E(r_c) = r_f + \sigma_c \dfrac{[E(r_p) - r_f]}{\sigma_p}$

$$\underbrace{E(r_c)}_{y} = \underbrace{r_f}_{a} + \underbrace{\sigma_c}_{x} \underbrace{\frac{E(r_p) - r_f}{\sigma_p}}_{b}$$

Equation of the straight line (refresh)

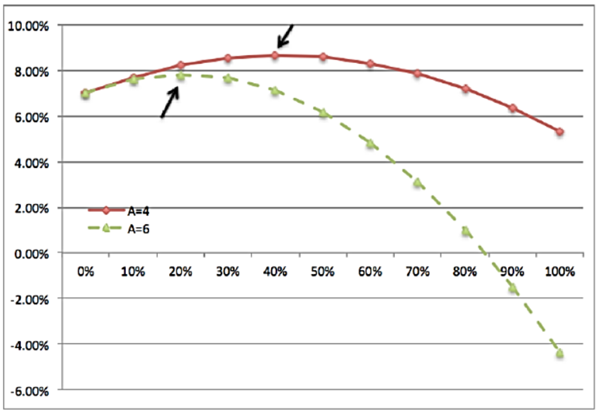
$$y = a + bx \qquad b = \frac{y_A - y_B}{x_A - x_B}$$





1.6 Optimal Portfolios

1. We have CAL: the graph of all feasible risk-return combinations available from different asset allocation choices
2. choose the optimal portfolio from the set of feasible choices based on personal preferences
3. capture preferences with the mean-variance utility function
4. the most preferred portfolio will be the one with the highest utility $max_{\omega}U$



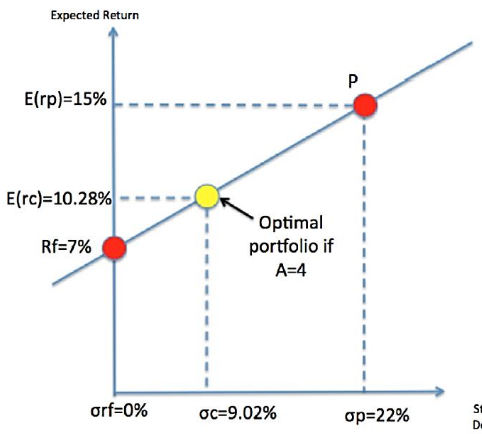
$$max_{\omega}U = \underbrace{r_f + \omega\left[E(r_p) - r_f\right]}_{E(r_c)} - \frac{1}{2}A\underbrace{(\omega^2\sigma_p^2)}_{\sigma_c^2}$$

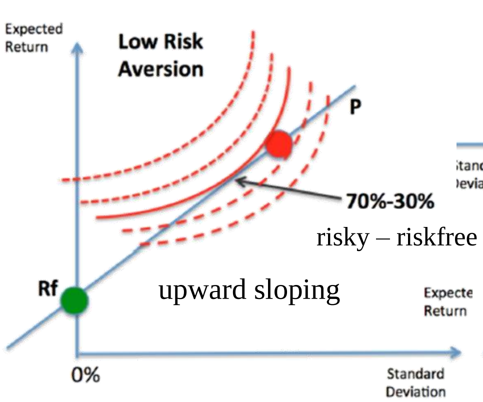
Set the derivative to zero:

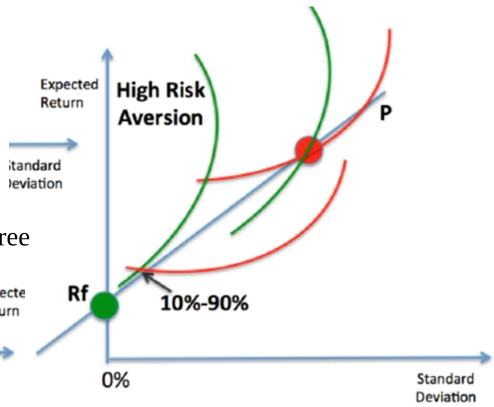
$$\frac{\delta U}{\delta \omega} = E(r_p) - r_f - A\sigma_p^2\omega = 0$$

$$\omega^* = \frac{E(r_p) - r_f}{A\sigma_p^2}$$ => indiviudal optimal portfolio

ω^* omega: the fraction of wealth invested in the risky asset.
$1 - \omega^*$: risk-free proportion



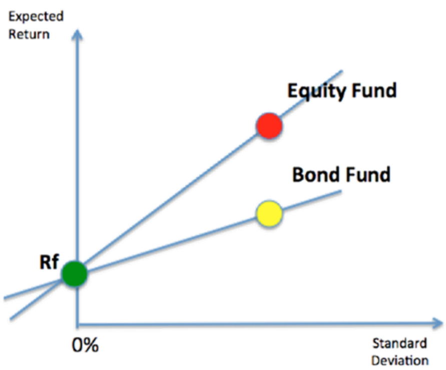


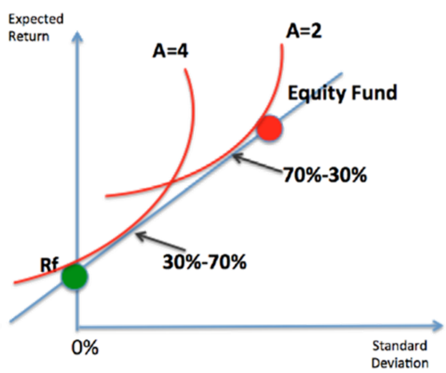


70%-30% is the individual investor's optimal portfolio
=> tangency with the indifference line with and CAL.

Acceptable to the

Available in the market





1.7 Practice of Asset Allocation

Reality check:

$$\omega^* = \frac{E(r_p) - r_f}{A\sigma_p^2}$$

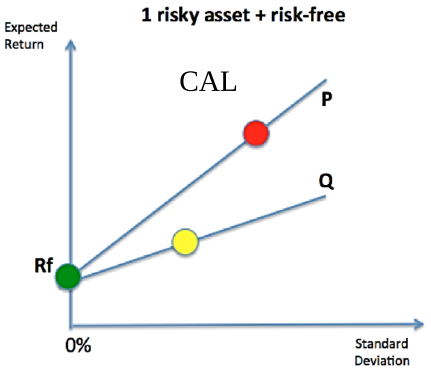
$$E(r_c) = r_f + \sigma_c \frac{\left[E(r_{equity}) - r_f\right]}{\sigma_{equity}}$$

$$E(r_c) = r_f + \sigma_c \frac{\left[E(r_{bond}) - r_f\right]}{\sigma_{bond}}$$

LECTURE 2 Asset Allocation

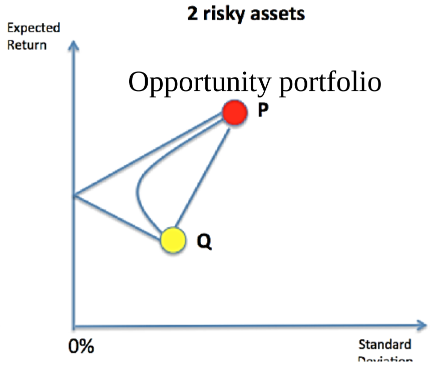
1. *Diversification* reduces risk
2. There are *limits*, not all risk can be diversified
   - Idiosyncratic risk is diversifiable
   - Systematic risk is not-diversifiable, cannot be cancelled
3. You do not need so many stocks to diversity risk
4. The lower the *correlation* the higher the reduction of risk achieved with diversification
5. Only with the power of math we can establish that
   - $\rho = 1$, no benefit from diversification $\sigma_p \geq \min(\sigma_D, \sigma_E)$
   - $\rho < 1$, diversification reduces risk $\sigma_p < \min(\sigma_D, \sigma_E)$
   - $\rho = -1$, all risk is completely diversified: $\sigma_p = 0$
6. When 2 risky assets and risk-free rate, we choose the best portfolio obtained combining the risk-free and the tangency portfolio (i.e. best portfolio on *CAL(T)*)

$$max_{\omega_D} U = \underbrace{\omega_D E(r_D) + (1-\omega_D)E(r_E)}_{E(r_p)} - \frac{1}{2}A\underbrace{(\omega_D^2\sigma_D^2 + \omega_E^2\sigma_E^2 + 2\omega_D\omega_E\sigma_{D,E})}_{\sigma_p^2}$$



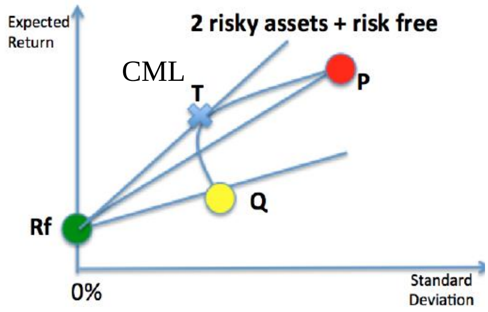
Lec1: *Capital allocation* b/w 1 risky portfolio and the risk-free asset

$$r_c = \omega r_p + (1-\omega)r_f$$
$$\sigma_c^2 = \omega^2\sigma_p^2$$



Lec2: *Asset allocation* across 2 risky asset classes

$$E(r_p) = \omega_D E(r_D) + \omega_E E(r_E)$$
$$\sigma_p^2 = \omega_D^2\sigma_D^2 + \omega_E^2\sigma_E^2 + 2\omega_D\omega_E\sigma_{D,E}$$



Lec2: *Asset allocation* b/w 2 risky asset classes plus the risk-free asset

2.1 Diversification

The limits of diversification

- Cannot cancel risk completely => *not-diversifiable risk (systematic risk)*
  - e.g. Risk that economy is in recession, then aggregate consumption drops: people buy neither of two stocks

Systematic and Idiosyncratic risk

- Risk that affects the entire *world* (World war III, 911, natural disaster)
- Risk that affects individual *countries* (inflation, exchange rate, interest rate)
- Risk that affects individual *industries* (technological innovations, changes in regulations, e.g. Uber)
- Risk that affects individual *firms* (death of CEO, new drug discover)

$$TotalRisk = \underbrace{SystematicRisk}_{not-diversifiable} + \underbrace{IdiosyncraticRisk}_{diversifiable}$$

- Examples: invest in several
  - Stocks in same country, same industry —> wash away firm idiosyncratic

$$TotalRisk = \underbrace{SystematicRisk}_{world/country/industry} + \underbrace{IdiosyncraticRisk}_{firm}$$

  - Stocks in same country, different industry => wash away industry and firm idiosyncratic

$$TotalRisk = \underbrace{SystematicRisk}_{world/country} + \underbrace{IdiosyncraticRisk}_{industry/firm}$$

  - Stocks in different country, different industry => wash away country, industry and firm idiosyncratic

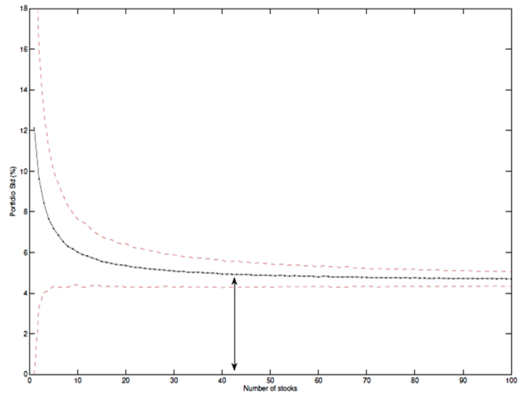
$$TotalRisk = \underbrace{SystematicRisk}_{world} + \underbrace{IdiosyncraticRisk}_{country/industry/firm}$$

It also depends on constraints

- Portfolio managers have specific mandates 指令
- Individual investors suffer transaction costs
- Individual investors have cognitive biases 认知偏差

How many stocks needed to make a diversified portfolio?

- *Not many, about 30 stocks, b/c stocks are cross-correlated*
- Systematic risk of US market is about 5.5% (monthly)



<u>2.2 Covariance & Correlation</u>

The benefits of diversification

The benefits of diversification are closely related to how stocks *covary*

Covariance $\sigma_{x,y}$

- it's a measure of how much two random variables change together

x	↑	↓	↓	↑	↑	↓	↑↓	↑↓
y	↑	↓	↑	↓	↑↓	↑↓	↑	↓
covar	+	+	-	-	0	0	0	0

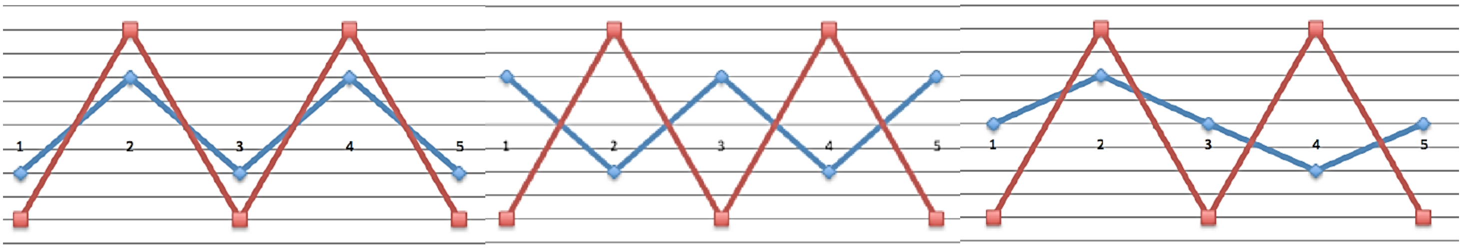
$$Cov(x,y) = \sigma_{x,y} = E[(x - E(x))(y - E(y))] = \sum_{s=1}^{N} p_s[(x_s - E(x))(y_s - E(y))]$$

- *intuition:* is x above its average when y is about its average
- *sample counterpart*

$$\widehat{Cov}(x,y) = \hat{\sigma}_{x,y} = \frac{1}{N}\sum_{s=1}^{N}(x_s - \bar{x})(y_s - \bar{y})$$, positive only if derivations have same sign

If equally possible, $Cov(x,y) = \frac{1}{N}\sum P_s$

- Property of covariance $Cov(ax, bk) = abCov(x,y)$, property of variance $Var(kx) = k^2 Var(x)$
- $Cov(x,k) = 0$? where k is constant, x is a random variable



Positive covariance $Cov > 0$ Negative covariance $Cov < 0$ Zero covariance $Cov = 0$

Coefficient of correlation $\rho_{x,y}$s

$$Correlation\ (x,y) = \rho_{x,y} = \frac{\sigma_{x,y}}{\sigma_x \sigma_y}$$ $(-1 \le \rho \le 1)$ interpret both magnitude and size

The *lower* the correlation ($\rho = -1$), the *higher* the reduction of risk achieved with diversification

Covariance Matrix

- Covariance matrix

$$\begin{bmatrix} \sigma_1^2 & \sigma_{1,2} & \cdots & \sigma_{1,N-1} & \sigma_{1,N} \\ \sigma_{2,1} & \sigma_2^2 & \cdots & \sigma_{2,N-1} & \sigma_N \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{N-1,1} & \sigma_{N-1,2} & \cdots & \sigma_{N-1}^2 & \sigma_{N-1,N} \\ \sigma_{N,1} & \sigma_{N,2} & \cdots & \sigma_{N,N-1} & \sigma_N^2 \end{bmatrix}$$

- Variances, on the diagonal
- Covariances, off-diagonal ($\sigma_{i,j} = \sigma_{j,i}$)

You don't need to make sure the correlation is negative to achieve the diversification. As long as the total portfolio variance is less than the minimum of either asset variance, $\sigma_p < \min(\sigma_D, \sigma_E)$, the diversification reduces risk. Even if the correlation is positive, the portfolio standard deviation STILL is LESS than the weighted average of the individual security standard deviations (unless the two securities are perfectly positively correlated $\rho = 1$)

2.3 Risky Assets

Portfolio of two risky assets

Efficient diversification: optimal weights that provide the lowest possible risk

- Optimal portfolio has *less variance*

Two risky assets: Equity fund (E) / Bond fund (D)

- Return: $r_p = \omega_D r_D + \omega_E r_E$ (weighted average of two returns)
- Expected return: $E(r_p) = \omega_D E(r_D) + \omega_E E(r_E)$ (weighted average of expected returns)
- Variance: $\sigma_p^2 = \omega_D^2\sigma_D^2 + \omega_E^2\sigma_E^2 + 2\omega_D\omega_E\sigma_{D,E}$ (*NOT the weighted sum of variances* => The variance of the portfolio is a *weighted sum of covariances*, and each weight is the product of the portfolio proportions of the pair of assets in the covariance term.)

B. Border-Multiplied Covariance Matrix

Portfolio Weights	w_D	w_E
w_D	$w_Dw_D\text{Cov}(r_D, r_D)$	$w_Dw_E\text{Cov}(r_D, r_E)$
w_E	$w_Ew_D\text{Cov}(r_E, r_D)$	$w_Ew_E\text{Cov}(r_E, r_E)$
$w_D + w_E = 1$	$w_Dw_D\text{Cov}(r_D, r_D) + w_Ew_D\text{Cov}(r_E, r_D)$	$w_Dw_E\text{Cov}(r_D, r_E) + w_Ew_E\text{Cov}(r_E, r_E)$
Portfolio variance	$w_Dw_D\text{Cov}(r_D, r_D) + w_Ew_D\text{Cov}(r_E, r_D) + w_Dw_E\text{Cov}(r_D, r_E) + w_Ew_E\text{Cov}(r_E, r_E)$	

Each covariance has been multiplied by the weights from the row and the column in the borders. General formula $\sigma_p^2 = \Sigma_{i=1}^N \Sigma_{j=1}^N \omega_i\omega_j\sigma_{i,j}$

Why $var(kx) = k^2 var(x)$ *?*

$$Var(x) = E((x-\bar{x})^2) = \sum_{s=1}^{S} p_s(x_s-\bar{x})^2$$

$$Var(kx) = \sum_{s=1}^{S} p_s(kx_s-k\bar{x})^2 = \sum_{s=1}^{S} p_s(k(x_s-\bar{x}))^2 = k^2\sum_{s=1}^{S} p_s(x_s-\bar{x})^2 = k^2 var(x)$$

Covariance of a random variable with itself, is its variance.

$$Cov(D,D) = \sum_{s=1}^{S} p_s[(r_{s,D}-E(r_D))(r_{s,D}-E(r_D))] = \sum_{s=1}^{S} p_s[(r_{s,D}-E(r_D))^2] = \sigma_D^2$$

Benefits from diversification if $\rho < 1$

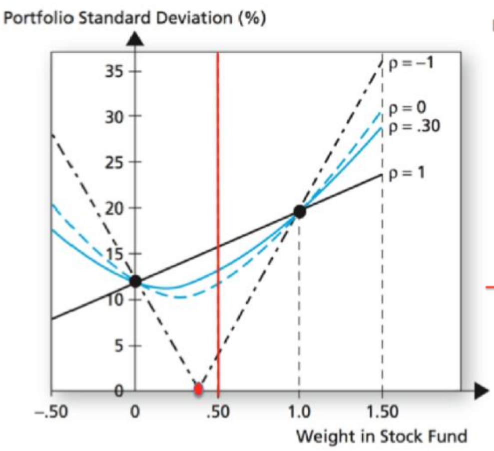
Even if the correlation is positive, the portfolio standard deviation STILL is LESS than the weighted average of the individual security standard deviations (unless the two securities are perfectly positively correlated $\rho = 1$)

$$\sigma_p^2 = \omega_D^2\sigma_D^2 + \omega_E^2\sigma_E^2 + 2\omega_D\omega_E\sigma_{D,E}$$
$$\sigma_p^2 = \omega_D^2\sigma_D^2 + \omega_E^2\sigma_E^2 + 2\omega_D\omega_E\sigma_D\sigma_E\rho_{DE}$$

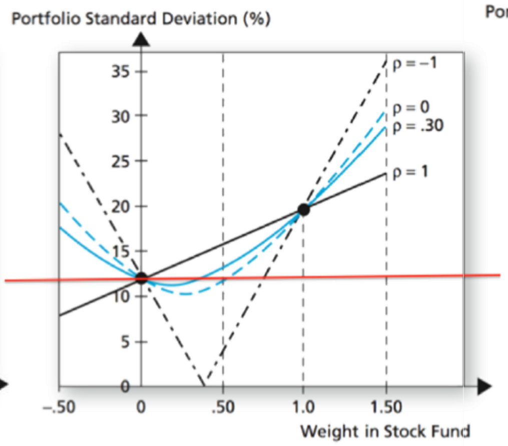
- If $\rho = 1$, no benefit from diversification, $\sigma_p \geq \min(\sigma_D, \sigma_E)$

  $\sigma_p^2 = \omega_D^2\sigma_D^2 + \omega_E^2\sigma_E^2 + 2\omega_D\omega_E\sigma_D\sigma_E(1)$ [Right-hand side is a perfect square $(a+b)^2 = a^2 + b^2 + 2ab$]

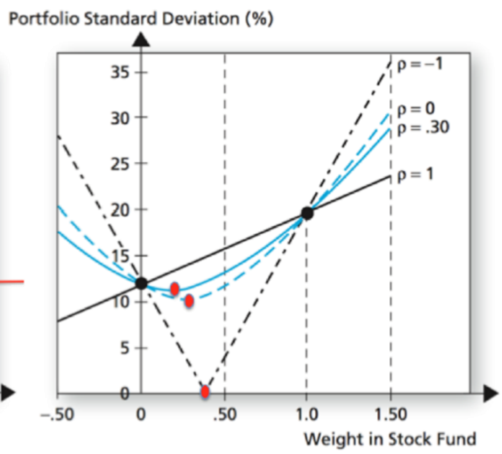
  $\sigma_p = \sqrt{(\omega_D\sigma_D + \omega_E\sigma_E)^2} = \omega_D\sigma_D + \omega_E\sigma_E$ [leaner function of the assets' weight]
- If $\rho < 1$, diversification reduces risk $\sigma_p < \min(\sigma_D, \sigma_E)$ 曲线
- If $\rho = -1$, perfect diversification $\sigma_p = 0 \ll \min(\sigma_D, \sigma_E)$ 折线



$0 < \omega_E < 1$, here $\omega_E = 0.5$, $\rho = -1$, always the lowest portfolio standard deviation, perfect diversification on D&E



Given the same portfolio standard deviation, the larger ω_E (the riskier asset in a set) requires the lower ρ



Plot the minimum variance upon each ρ line. Which is the best? Min variance portfolio

Minimum Variance Portfolio

$$min_{\omega_D}\sigma_p^2 = \omega_D^2\sigma_D^2 + \omega_E^2\sigma_E^2 + 2\omega_D\omega_E\sigma_{D,E}$$
$$= (1-\omega_D)^2\sigma_E^2 + \omega_D^2\sigma_D^2 + 2(1-\omega_D)\omega_D\sigma_{D,E}$$

Set the derivative to zero

$$\frac{\delta\sigma_p^2}{\delta\omega_D} = -2(1-\omega_D)\sigma_E^2 + 2\omega_D\sigma_D^2 + 2\sigma_{D,E} - 4\omega_D\sigma_{D,E} = 0$$

$$\rightarrow \omega_D\left(\sigma_E^2 + \sigma_D^2 - 2\sigma_{D,E}\right) = \sigma_E^2 - \sigma_{D,E}$$

$$\omega_D^* = \frac{\sigma_E^2 - \sigma_{D,E}}{(\sigma_E^2 + \sigma_D^2 - 2\sigma_{D,E})}$$ [in formula sheet]

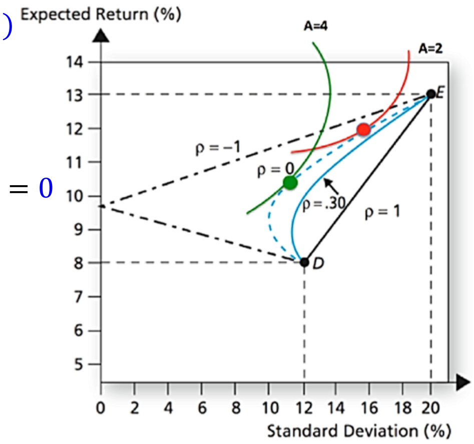
$$\omega_E^* = 1 - \omega_D^*$$

$$\sigma_p^{*2} = \omega_D^{*2}\sigma_D^2 + \omega_E^{*2}\sigma_E^2 + 2\omega_D^*\omega_E^*\sigma_{D,E}$$

<u>**2.4 Solution**</u>

Set of feasible portfolios

$$max_{\omega_D}U = \underbrace{\omega_D E(r_D) + (1-\omega_D)E(r_E)}_{E(r_p)} - \frac{1}{2}A(\underbrace{\omega_D^2\sigma_D^2 + \omega_E^2\sigma_E^2 + 2\omega_D\omega_E\sigma_{D,E}}_{\sigma_p^2})$$



Set the derivative to zero

$$\frac{\delta U}{\delta\omega_D} = E(r_D) - E(r_E) - \frac{1}{2}A\underbrace{[-2(1-\omega_D)\sigma_E^2 + 2\omega_D\sigma_D^2 + 2\sigma_{D,E} - 4\omega_D\sigma_{D,E}]}_{\frac{\delta\sigma_p^2}{\delta\omega_D}} = 0$$

$$\rightarrow E(r_D) - E(r_E) - A[\omega_D\left(\sigma_E^2 + \sigma_D^2 - 2\sigma_{D,E}\right) - \sigma_E^2 + \sigma_{D,E}] = 0$$

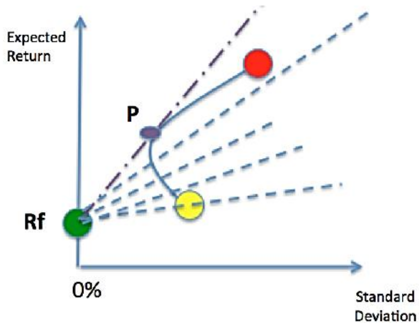
$$\omega_D^* = \frac{E(r_D) - E(r_E) + A[\sigma_E^2 - \sigma_{D,E}]}{A(\sigma_E^2 + \sigma_D^2 - 2\sigma_{D,E})}$$ [not in formula sheet]

<u>**2.5 Two risky + risk-free**</u>

Best feasible CAL function

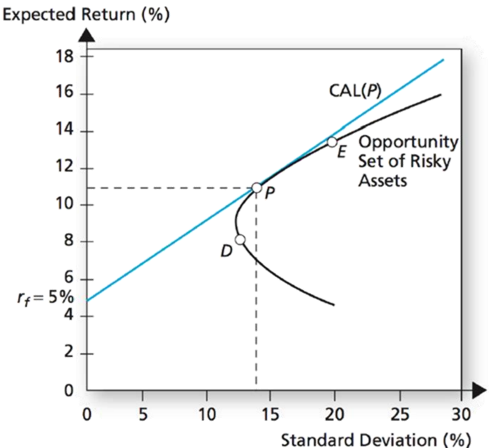
$$\underbrace{E(r_C)}_{DepVariable:y} = \underbrace{r_f}_{intercept:a} + \underbrace{\sigma_C}_{DepVariable:x}\underbrace{\frac{E(r_P) - r_f}{\sigma_P}}_{slope:SharpeRatio}$$

The best feasible CAL is the one with the *highest* slop (sharpe ratio)



Until the point of *tangency* with the investment opportunity set: CAL with the highest slope and so the one that *dominates* all CALs

> CAL vs CML?
> CAL is the curve connecting an individual risky portfolio (A, B…) and riskfree asset.
> CAL is 1 risky + 1 riskfree
> CML tangents the market portfolio (point M), special case of CAL,
> CML is a passive investment strategy (market portfolio)



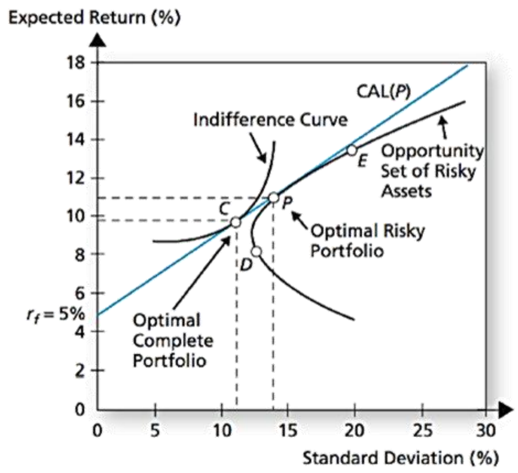
Find tangency P (optimal weight for each asset ω_D, ω_E?)

$$max_{\omega_D}Sharpe = \frac{\overbrace{\omega_D E(r_D) + (1-\omega_D)E(r_E)}^{E(r_p)} - r_f}{\underbrace{\sqrt{\omega_D^2\sigma_D^2 + \omega_E^2\sigma_E^2 + 2\omega_D\omega_E\sigma_{D,E}}}_{\sigma_P}}$$ [not in formula sheet]

$$\omega_D = \frac{E(r_D - r_f)\sigma_E^2 - E(r_E - r_f)\sigma_{D,E}}{E(r_D - r_f)\sigma_E^2 + E(r_E - r_f)\sigma_D^2 - [E(r_D - r_f) + E(r_E - r_f)]\sigma_{D,E}}$$

$$w_E = 1 - w_D$$

<u>Fisher's separation theory:</u> a firm's choice of investments is separate from its owner's attitudes towards the investments (irrelevant to A). it's possible to separate a firm's investment form financial decisions.



1. opportunity set (b/w 2 risky assets)

$$E(r_p) = \omega_D E(r_D) + \omega_E E(r_E)$$
$$\sigma_p^2 = \omega_D^2 \sigma_D^2 + \omega_E^2 \sigma_E^2 + 2\omega_D \omega_E \sigma_{D,E}$$

2. Covariance Matrix

B. Border-Multiplied Covariance Matrix

Portfolio Weights	**w_D**	**w_E**
w_D	$w_D w_D \mathrm{Cov}(r_D, r_D)$	$w_D w_E \mathrm{Cov}(r_D, r_E)$
w_E	$w_E w_D \mathrm{Cov}(r_E, r_D)$	$w_E w_E \mathrm{Cov}(r_E, r_E)$
$w_D + w_E = 1$	$w_D w_D \mathrm{Cov}(r_D, r_D) + w_E w_D \mathrm{Cov}(r_E, r_D)$	$w_D w_E \mathrm{Cov}(r_D, r_E) + w_E w_E \mathrm{Cov}(r_E, r_E)$
Portfolio variance	$w_D w_D \mathrm{Cov}(r_D, r_D) + w_E w_D \mathrm{Cov}(r_E, r_D) + w_D w_E \mathrm{Cov}(r_D, r_E) + w_E w_E \mathrm{Cov}(r_E, r_E)$	

Each covariance has been multiplied by the weights from the row and the column in the borders. General formula $\sigma_p^2 = \Sigma_{i=1}^{N} \Sigma_{j=1}^{N} \omega_i \omega_j \sigma_{i,j}$

3. the most preferred portfolio (opportunity portfolio)

$$max_{\omega_D} U = \underbrace{\omega_D E(r_D) + (1 - \omega_D) E(r_E)}_{E(r_p)} - \frac{1}{2} A \underbrace{(\omega_D^2 \sigma_D^2 + \omega_E^2 \sigma_E^2 + 2\omega_D \omega_E \sigma_{D,E})}_{\sigma_p^2}$$

$$\omega_D^* = \frac{E(r_D) - E(r_E) + A[\sigma_E^2 - \sigma_{D,E}]}{A(\sigma_E^2 + \sigma_D^2 - 2\sigma_{D,E})}$$

4. find minimum variance portfolio (b/w 2 risky assets)

$$\omega_D^* = \frac{\sigma_E^2 - \sigma_{D,E}}{(\sigma_E^2 + \sigma_D^2 - 2\sigma_{D,E})}$$
$$\omega_E^* = 1 - \omega_D^*$$
$$E(r_p^*) = \omega_D^* E(r_D) + \omega_E^* E(r_E)$$
$$\sigma_p^{*2} = \omega_D^{*2} \sigma_D^2 + \omega_E^{*2} \sigma_E^2 + 2\omega_D^* \omega_E^* \sigma_{D,E}$$

5. find optimal risky set (b/w 2 risky assets)

$$\omega_D = \frac{E(r_D - r_f)\sigma_E^2 - E(r_E - r_f)\sigma_{D,E}}{E(r_D - r_f)\sigma_E^2 + E(r_E - r_f)\sigma_D^2 - [E(r_D - r_f) + E(r_E - r_f)]\sigma_{D,E}}$$
$$w_E = 1 - w_D$$

$$max_{\omega_D} Sharpe = \frac{\overbrace{\omega_D E(r_D) + (1 - \omega_D) E(r_E)}^{E(r_p)} - r_f}{\underbrace{\sqrt{\omega_D^2 \sigma_D^2 + \omega_E^2 \sigma_E^2 + 2\omega_D \omega_E \sigma_{D,E}}}_{\sigma_p}}$$

6. find CAL (2 risky + 1 riskfree)

$$E(r_c) = r_f + \sigma_c \frac{E(r_p) - r_f}{\sigma_p}$$
$$y = a + x \cdot SharpeRatio$$