Topic 1: Introduction to Bond Pricing

Wednesday, 27 July 2016
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What is a bond?

- Borrowing/lending contract with three key parameters: principal, maturity, interest rate
- A bond is a **claim on some fixed future cash flows**, typically with the bond's face value payable at maturity
- Smaller cash flows before maturity are known as coupons, usually expressed as a percentage of the face value **(coupon rate)**
- Bonds contain some level of **default risk** (risk that borrower will not be able to pay lender)
- Other common assumptions: no transaction costs, constant interest rates, complete markets, etc.

- Two approaches to pricing a bond:
  - **Fundamental pricing:** using demand and supply for the asset to determine price
  - **Arbitrage pricing:** replicating an asset's future cash flows with a portfolio of assets with known prices to determine its price

What is arbitrage?

- Arbitrage is a set of trades generating a **positive and risk free cash flow today** with zero cash flows in the future
  - Effectively free money
- An example is a **violation of the law of one price**, e.g. a bond selling for two different prices (buy low sell high for instant profit)

Arbitrage pricing

- Arbitrage pricing involves creating a **replicating portfolio**:
  - e.g. to price a risk free one year zero-coupon bond, we would make a synthetic version of the bond, i.e. an investment that mimics its cash flows exactly
  - For example we could a particular amount of money, M in the bank at a known interest rate (provided the investment has the same amount of risk) to receive $100 in a year's time
  - Therefore according to arbitrage pricing, our bond should be exactly equal to M (which is the $100 discounted at the interest rate)
- If the bond price differs from the price given based on this method, e.g. if it was lower, one could profit off arbitrate trading
  - Buy borrowing money from the bank and using it to buy a bond, all **future cash flows would cancel out** to zero but a risk free profit would be made today
  - In practice, traders identity arbitrage profits and take advantage of them, **increasing demand for the lower priced** one and **increasing supply of the higher priced** one until the **price mechanism makes these two prices identical**

Arbitrage pricing: general case (Cont.)

- Replicating a cash flow of c_t at time t
  - Today, deposit M_t such that $M_t(1 + y_t)^t = c_t$
  - Find that $M_t = c_t / (1 + y_t)^t$
- Replicating the cash flow at maturity, time T
  - Today, deposit M_T such that $M_T(1 + y_T)^T = FV + c_T$
  - Find that $M_T = (FV + c_T)/(1 + y_T)^T$
- Our complete strategy costs – which should equal to price

$$P = M_1 + M_2 + \ldots + M_t + \ldots + M_T = c_1 / (1 + y_1)^1 + c_2 / (1 + y_2)^2 + \ldots + c_t / (1 + y_t)^t + \ldots + (FV + c_T)/(1 + y_T)^T$$

Discounting

- Calculating the PV of future CFs is called discounting
- The rate at which we discount a cash flow is **determined by the market** according to its associated level of risk (and technically a liquidity premium)
- The price of a bond is the **sum of the present values of its future cash flows**, where demand and supply influences the appropriate discount rate

Pricing formula and yield-to-maturity

- Assuming constant interest rates:

$$P = c_1 / (1 + y)^1 + c_2 / (1 + y)^2 + \ldots + c_t / (1 + y)^t + \ldots + c_T/(1 + y)^T + FV/(1 + y)^T$$

- Interest rates are not constant in practice, but we take P as given and define y as whatever interest rate satisfies the above,
  - i.e. **YTM summarises all future interest rates** determining the bond price by kind of averaging it into one
- $P = \frac{c}{y}\left(1 - \frac{1}{(1+y)^T}\right) + \frac{FV}{(1+y)^T}$
- **If the YTM increases, the bond price decreases** (since a higher return is demanded for the same face value)
  - C% = YTM: bond trades at par, P = FV
  - C% < YTM: bond trades at a discount, P < FV
  - C% > YTM: bond trades at a premium, P > FV
- YTM does not usually equal to realised holding period returns because:
  - The YTM will change over time so the realised yield would be somewhere in between the extremes
  - **Intermediate cash flows may be reinvested** until maturity at a rate different to the YTM

Realised compound yield

- For example, to compare the yields of a zero coupon bond with a coupon bond, the coupon will have to be reinvested to make cash flows at maturity comparable
- To calculate, just assume all cash flows are collected at maturity and use the $(FV/PV)^{1/n}$ - 1 to calculate annual return
- The realised compound yield is the YTM **only if the coupons are reinvested at the YTM rate**
    - It follows that if the coupon reinvestment rate is less than the YTM, our realised compound yield would be lower
- Reinvestment rates are determined by the market and realised in the future

Topic 2: Term Structure of Interest Rates

Thursday, 4 August 2016
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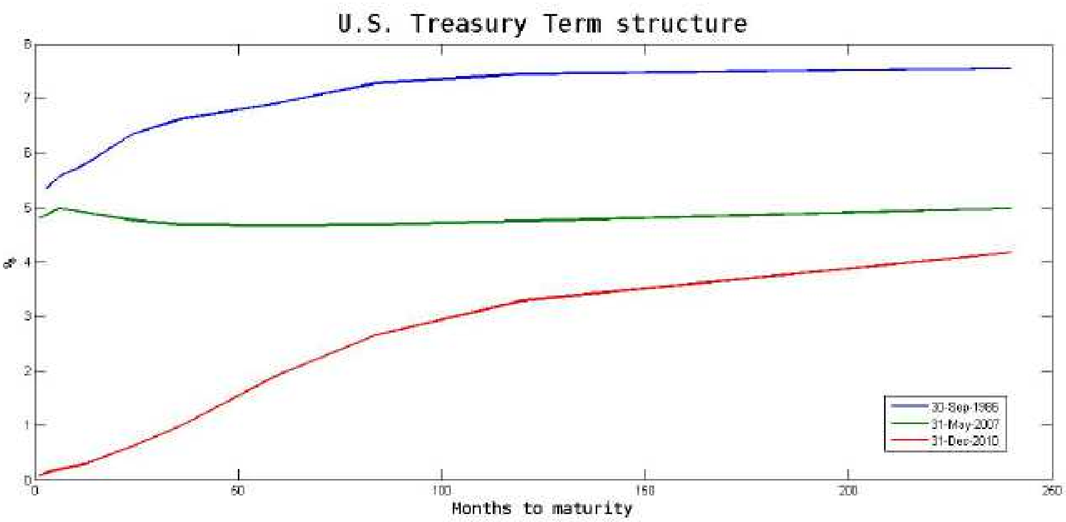
Introduction

- Discount rates for future cash flows to be realised in different times are determined by the market
- y(t) does differ across different horizons, and this is known as the **term structure of interest rates**
- t-period spot rate (y_t)
    - Is the interest rate today for a t-period investment which **does not have any cash flow over the investment period** except for one at maturity, informally known as the **zero rate**
- t-period spot rate is different to YTM of a t-period bond
    - difference between y and y_t - yield to maturity is found as the variable y which fits the equation C/(1+y) + C/(1+y)^2 + …
    - Spot rate is known as **pure yield**

Term structure

- The term structure can be seen as the relation between spot rates and maturity horizons, i.e. between y_t and t



- So what determines the yield curve? There are a few hypotheses
- The spot rates and term structures are implied by market bond prices
    - Inferring spots from zero coupon bonds:

- $$P = \frac{FV}{(1+y_t)^t} \rightarrow y_t = \left(\frac{FV}{P}\right)^{\frac{1}{t}} - 1$$
  - Inferring spots from coupon bonds:
    - Get the pricing equation with all the unknowns for each y_t
    - Find y_1 from a one-period zero coupon bond, then substitute into the two-period equation to find y_2, then substitute into the three-period equation to find y_3 and so on
- If the prices of different bonds imply **inconsistent spot rates, there are arbitrage opportunities**
  - A trader can replicate the bond's cash flows with a different bond in the market for a different price (buy low, sell high, cancel out future cash flows for immediate risk-free profit)
- To set up the arbitrage trades, it is possible to **construct a synthetic version of a bond** with two other bonds provided cash flow times allow it, working out the proportions required of the two bonds
  - Out of the synthetic and actual bonds, buy the low one and sell the high one
- According to the **market expectations hypothesis**, an investor can expect the same return from securities which are identical except for their length of maturity
  - i.e. investing in a one year bond, and then in one year using those profits to purchase another one year bond, should give an investor the same return as if he had invested in the same two-year bond
- This means that long term interest rates would be calculated based on all intermediate short term interest rates
  - However since the world is not perfect, they are theorised to be based off **expected short term rates**

Forward rates

- Forward rates are interest rates for investments set at a **predetermined rate for a predetermined time in the future**
- Forward rate determined today for an investment starting at time *s* ending at time *t* is denoted ${}_sf_t$
- How to find ${}_{t-1}f_t$
  - $(1+y_t)^t = (1+y_{t-1})^{t-1}(1+{}_{t-1}f_t)$
  - Rearranging: ${}_{t-1}f_t = (1+y_t)^t / (1+y_{t-1})^{t-1} - 1$
- Therefore to find ${}_sf_t$
  - $(1+y_t)^t = (1+y_s)^s(1+{}_sf_t)^{t-s}$
  - Rearranging:

$${}_sf_t = [(1+y_t)^t / (1+y_s)^s]^{1/(t-s)} - 1$$

Expectations hypothesis

- In reality, we don't know ${}_1y_2$ at t = 0, so the equality probably won't hold
- Instead the market has expectations about ${}_1y_2$ so we use that instead
  - $(1+y_2)^2 = (1+y_1)*(1+E({}_1y_2))$
- The **expectations hypothesis** states that **market expectations on future interest rates determine spot rates** over different horizons and the term structure
  - In other words, ${}_sf_t = E({}_sy_t)$

Liquidity preference hypothesis

- In reality, bond issuers and investors have different horizons, i.e. some prefer short term bonds, others prefer long term bonds
- If we have an investment horizon of one year, we can either invest in a 1-year zero coupon bond or hold a 2-year zero and sell it at t=1
    - The first option will guarantee our holding period returns as y_1, but our option 2 will depend on ${}_1y_2$
    - This means our option 2, i.e. the longer term bond, has **liquidity risk**
- Also, if we have an investment horizon of 2 years, we can either (i) invest in a two-year zero or (ii) hold a 1-year zero and reinvest it at t=1
    - Holding period returns after the two years:
        - (i) guaranteed as $(1+y_2)^2 - 1$
        - (ii) $(1+y_1)*(1+{}_1y_2) - 1$
    - Since we don't know ${}_1y_2$, option 2 is risky, i.e. the shorter term bond has **reinvestment risk**
- If issuers typically prefer to issue long-term bonds, they have **longer investment horizons** than investors
    - Short-term investors would need to carry the **liquidity risk** of holding long-term bonds, and there would be a **liquidity premium** to incite investors to hold long-term
    - This results in higher yields for long-term bonds and the upward-sloping normal curve
- Technically the effect could go both ways, depending on what issuers prefer and how it differs to investor preferences
    - When investors have longer investment horizons than issuers, they need to carry reinvestment risk to hold short-term investments so the curve would be downward sloping

- So which theory works? In practice, probably both contribute so that:
- $_sf_t = E(_sy_t) + L$, where L is the liquidity premium

Topic 3: Duration

Friday, 12 August 2016
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Interest rate risk in bonds

- Since, usually, an increase in interest rates will decrease bond prices, and a decrease in rates increases bond prices, bonds (and other securities) are **exposed to the risk of interest rate changes**
- So how do we measure this level of interest risk? By the **sensitivity** of **a security's market value** to **interest rate changes**
- Historically, interest rate risk can, somewhat ineffectively, be measured by **maturity**, since bonds with **higher maturities are more sensitive to rate changes**
- Plotting price on interest rate shows that the **price curve is convex**, i.e. decreases in the interest rate affect the price more than equally sized increases
- Deriving the bond price formula to determine the rate of change for price based on yield:

- $$P = \sum_{t=1}^{T} \frac{CF_t}{(1+y)^t}$$
- $$\frac{\partial P}{\partial y} = \frac{\partial\left(\sum_{t=1}^{T} \frac{CF_t}{(1+y)^t}\right)}{\partial y}$$
- $$= \frac{\sum_{t=1}^{T} \partial\left(\frac{CF_t}{(1+y)^t}\right)}{\partial y}$$
- $$= \sum_{t=1}^{T} CF_t(-t)\frac{1}{(1+y)^{t+1}}$$
- $$= -\sum_{t=1}^{T} \frac{CF_t}{(1+y)^{t+1}} \times t$$

Duration

- Since maturity is a bit ambiguous, we use **duration**, which effectively measures the **weighted length of time before cash flows are received**, i.e. average maturity of cash flows weighted by size of cash flows (at their present value)
- Duration (specifically**, Macaulay duration**) is also the **relative change in price given a 1% relative change in the yield** (makes sense after a lot of algebra trust me)

- $$D = \frac{\partial P / P}{\partial y / (1+y)}$$
    - $$= \frac{1+y}{P} \times \frac{\partial P}{\partial y}$$
    - $$= \frac{1+y}{\sum_{t=1}^{T} PV(CF_t)} \times -\sum_{t=1}^{T} \frac{CF_t}{(1+y)^{t+1}} \times t$$
    - $$= \frac{1}{\sum_{t=1}^{T} PV(CF_t)} \times -\sum_{t=1}^{T} \frac{CF_t}{(1+y)^{t}} \times t$$
    - $$= -\sum_{t=1}^{T} \frac{PV(CF_t)}{\sum_{t=1}^{T} PV(CF_t)} \times t$$

- $$D = \sum_{t=1}^{T} t \times w_t$$
    - $$w_t = \frac{PV(CF_t)}{\sum_{t=1}^{T} PV(CF_t)}$$

Interpretations

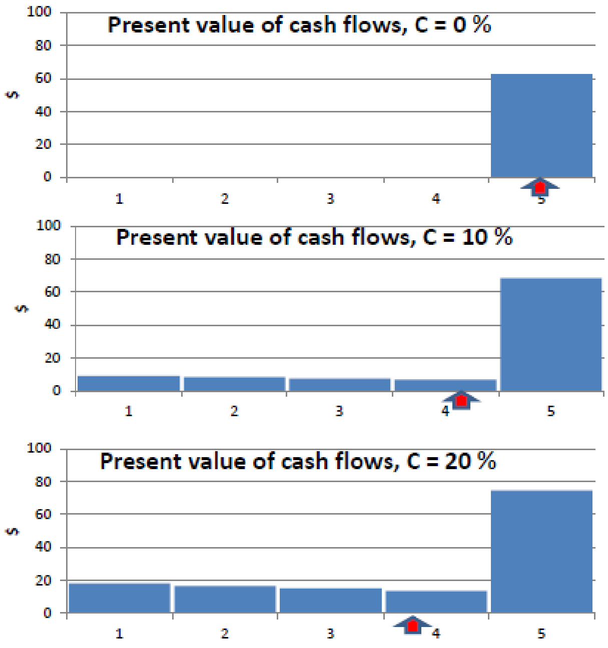
- Economic meaning
    - **Interest risk measure:** sensitivity of market value change to change in interest rate
- Maturity measure
    - **Average life of a bond**
    - Taking into account maturity of all cash flows but with appropriate weightings
    - Appropriate = dependent on contribution to present value of price
- Economic payback period
    - Measures how long **on average** it takes to get back cost of investment (PV amount)
- It is also the **gradient of the price-yield curve at that point**

Maturity and duration

- Duration **increases as maturity increases**,
    - which makes sense, since it would increase the average life of the security
- However it **increases at a decreasing rate**, since higher maturity cash flows have lower PVs and hence lower weightings

Coupon rate and duration

- Duration **decreases as coupon rate increases**
    - since it increases the weighting of cash flows closer to present than those further away



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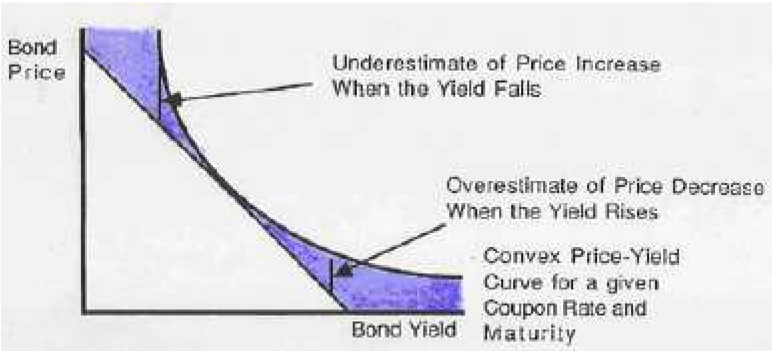
Yield and duration

- Duration **increases as yield decreases** and **decreases as yield increases**
- This is because changes in yield affect the higher maturity cash flows the most
  - Higher yield lowers PV of higher maturity cash flows and hence their weightings, lowering duration
  - Lower yield increases PV higher maturity cash flows and hence their weightings, increasing duration
- Also makes sense since **duration is the slope of the price-yield curve**, and the curve is convex (gradient gets shallower as yield increases)

Estimating price changes

- We can predict potential price changes based on duration, given a potential change in the interest rate
- $$\frac{\Delta P}{P} \approx -D \times \frac{\Delta y}{1+y}$$
  - Note: D* = D/(1+y) aka **modified duration**
- So why is it only approximately equal? Because when we use this equation, we are treating D as the gradient and moving along a **tangent to the price curve instead of the actual curve**
  - This means we will **underestimate price increases** and **overestimate price decreases**:



- In other words, the estimated price is always **smaller** than the actual price
- "Nature of the approximation error is the convex relationship between price and yield"

Convexity and price changes

- Price changes can be calculated more accurately (but still not exactly!) by taking into account the convexity of the price-yield curve
- $$\frac{\Delta P}{P} = \left(-D \times \frac{\Delta y}{1+y}\right) + \left(\frac{(\Delta y)^2}{2} \times convexity\right)$$
  - Pray you never need to calculate it, but formula for convexity is as follows:
    - $$convexity = \frac{1}{P \times (1+y)^2} \sum_{t=1}^{T} \left[\frac{CF_t \times (t^2 + t)}{(1+y)^t}\right]$$

Portfolio duration

- An entire portfolio of bonds can be seen as one huge aggregate bond to work out it's duration
- Duration of a portfolio is a **weighted average of its components' durations**