Topic 1: Linear Algebra

LINEAR SYSTEM

A set of simultaneous equations in which each variables occur to the power of 1.
No XY; X^2; Sin(x)

Example: 2x + y = 5 ← describes a line
Solution set of x=t ; y= 5-2t , parameter $t \in \mathbb{R}$
Can also be written as :

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ 5-2t \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} + t\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

TYPES OF 2 INTERSECTING LINES

- Non- intersecting and parallel (If this is the case, the system is inconsistent)
- Unique Intersection (System is consistent, there is an intersection point)
- Non-unique Intersection(System is consistent, there is an intersection point)

EQUATION

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{12}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ a_{m1}x_1 + a_{m2}x_{m2} + \cdots + a_{mn}x_n &= b_m \end{aligned}\right\}$$

Can be written as

$$Ax = b$$

where A is $m\ x\ n$ matrix
M is no. of rows (equations)
N is no. of columns (unknowns)

Put the above in augmented form [A|b]

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{1n} & b_1 \\ a_{21} & a_{22} \cdots & a_{2n} & b_2 \\ \vdots & \vdots & a_{mn} & b_m \end{array}\right]$$

RULES FOR ELEMENTARY ROW OPERATIONS (APPLIED TO AUGMENTED MATRIX)

1. Multiply row by a non-zero constant
2. Swap 2 rows
3. Add substract multiple of 1 row to another

Ex: $x + y + z = 2$
$2x + y = 4$
$x - y - z = 0$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & 0 & 4 \\ 1 & -1 & -1 & 0 \end{array}\right] \underset{R3 \to R3 - R1}{\overset{R2 \to R2 - 2R1}{\longmapsto}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -2 & 0 \\ 0 & -2 & -2 & -2 \end{array}\right] \underset{R3 \to R3 + R2}{\overset{R2 \to -R2}{\longmapsto}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & -2 \end{array}\right]$$

Becomes easy to solve as it's in triangular form. Called row echelon form.

Qualifications of Row Echelon Form:

1. The first non-zero number ("leading entry") in each non-zero row is 1, called a leading 1
2. Lower leading 1s appear to the right of the higher leading 1s (step formation)
3. All zero rows appear at the bottom of the matrix (entries below steps are 0)

Reduced Row Echelon Form

4. A column with a leading 1 consists of 0s everywhere else. The number of leading 1s in row echelon form is the RANK. Row echelon form of a matrix is not unique (lotsof solutions) but rank is well defined.

GAUSSIAN ELIMINATION

STEP 1: Interchange rows → so top of first column has a non-zero entry
STEP 2: Multiply first row by ≠0 number → so first non-zero term = 1 (leading 1)
STEP 3: Add multiple of top rows to lower rows → entry below leading 1 = 0
STEP 4: Repeat steps for all rows
ADDITIONAL STEP TO GET RREF (GAUSS-JORDAN ELIMINATION)
STEP 5: For each leading 1, add multiple of below rows to make entry above each leading 1 = 0
{Tips: Better to work backwards}

$$RANK + \#PARAMETERS = \#UNKNOWNS + \#COLUMNS$$

HOMOGENEOUS EQUATIONS → HAVE SOLUTION

- Ax = 0 is a unique solution if rank A = n (m x n matrix)
- Infinitely many solutions if rank A < n
- Rank is the number of non-zero rows in the row echelon form

INHOMOGENEOUS EQUATIONS → NO SOLUTION

- Rank A < Rank B [A|b]

IDEA OF PROOF

- **If B = RREF of A ; Ax = 0 has the same solution set as Bx = 0**
- **If Rank A = n = # of columns in A ; The RREF will have 1 leading entry in each column. → Meaning: Have unique solution**
- **If Rank B < n (# of columns in B) ; There will be columns with no leading entry. → Meaning: Presence of Parameters (# of unknowns) can solve uniquely for other parameters in terms of unknown. Have infinite solutions.**

PROPERTIES OF SCALAR MULTIPLICATION

$(a + b)C = aC + bC$
where a and b are scalars and c is the matrix
$a(B + C) = aB + aC$
$a(bC) = (ab)C$

Trace of square matrix is the sum of elements on the main diagonal

PROPERTIES OF MATRIX ADDITION

$A + (B + C) = (A + B) + C \;\cdots\cdots\cdots\; Associative\ Law$
$(A + B)^T = A^T + B^T$

Example: $Let\ A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}\ and\ B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}.\ Calculate\ AB\ and\ BA$

$$AB = \begin{bmatrix} 1(4) & 1(3) \\ 8+6 & 6+3 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 14 & 9 \end{bmatrix}$$

$$BA = \begin{bmatrix} 4(6) & 9 \\ 2+2 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 9 \\ 4 & 3 \end{bmatrix}$$

Same Trace! tr(AB) = tr(BA)