FINANCIAL MATHS I

Important equivalents
Some important equivalents of i and d,

$$v = 1 - d$$

$$(1-d)(1+i) = 1$$

$$\left(1 - \frac{d^{(p)}}{p}\right)\left(1 + \frac{i^{(p)}}{p}\right) = 1$$

$$1 + i = \left(1 + \frac{i^{(p)}}{p}\right)^p$$

$$1 - d = \left(1 - \frac{d^{(p)}}{p}\right)^p$$

1.4 Interest rates that vary over time

Effective rate of interest per time unit for period $(t, t+1)$ denoted as $i(t)$
→ Value of 1 at time t becomes $1 + i(t)$ at time $(t+1)$
→ Usually $i(t)$ is denoted as a form of function, for example $i(t) = 0.05 + 0.001t$

Nominal rate of interest $[i_h(t)]$, means that the effective rate of interest for a **period of length h**, starting at time **t** is $\boldsymbol{h \cdot i_h(t)}$**.**

Illustration
$i_{0.7}(0) = 0.08$,
This statement means that, the interest rate is **0.08** for **0.7**-time unit and the amount **1** at time **0** become $\mathbf{1 + (0.08 \cdot 0.7)}$ at time $\mathbf{(0 + 0.7)}$.

Accumulation
$A(t_1, t_2)$ is the accumulation at time t_2 of an investment **1** at time t_1

$$\therefore \boldsymbol{A(t, t+h) = 1 + h \cdot i_h(t)}$$

Rearranging formula above, we can conclude that,

$$\therefore \boldsymbol{i_h(t) = \frac{A(t, t+h) - 1}{h}}$$

Consistent market
If $t_1 < t_a < t_b < t_2$, then the following is true

$$\therefore \boldsymbol{A(t_1, t_2) = A(t_1, t_a) \cdot A(t_a, t_b) \cdot A(t_b, t_2)}$$

1.5 The force of interest

The force of interest at time t (denoted as $\delta(t)$) is the **limit** of **nominal interest rate** $\boldsymbol{i_h(t)}$ as the interval time **h tends to 0** from **above**.

$$\therefore \boldsymbol{\delta(t) = \lim_{h \to 0^+} [i_h(t)]}$$

As we know that from 1.4 that

$$i_h(t) = \frac{A(t, t+h) - 1}{h}$$

Assuming that it is in a consistent market,

$$\delta(t) = \lim_{h \to 0^+} \left[\frac{A(t, t+h) - 1}{h}\right]$$

$$\delta(t) = \frac{1}{A(0,t)} \lim_{h \to 0^+} \left[\frac{A(0, t+h) - A(0,t)}{h}\right]$$

Suppose $F(t)$ is the accumulated value at time t for investment at time 0

$$\delta(t) = \frac{1}{F(t)} \lim_{h \to 0^+} \left[\frac{F(t+h) - F(t)}{h}\right]$$

By the definition of derivatives,

$$\lim_{h \to 0^+} \left[\frac{F(t+h) - F(t)}{h}\right] = \frac{d}{dt}[F(t)] = F'(t)$$

$$\delta(t) = \frac{F'(t)}{F(t)} = \frac{d}{dr}[\log F(r)]$$

Therefore,

$$\therefore \boldsymbol{A(0, t) = F(t) = exp\left\{\int_0^t [F(r)] dr\right\}}$$

$$\therefore \boldsymbol{A(t, th) = exp\left\{\int_t^{t+h} [F(r)] dr\right\}}$$

$$\therefore \boldsymbol{1 + h \cdot i_h(t) = exp\left\{\int_t^{t+h} [F(r)] dr\right\}}$$

1.6 Present value with vary interest rates

$[A(0,t)]^{-1}$ is the PV of 1 due at time t, recall that this is the definition of v. Therefore,

$$\therefore \boldsymbol{v(t) = \exp\left\{-\int_t^{t+h} [F(r)] dr\right\}}$$