INTEGRAL CALCULUS

Riemann Sums

The Riemann sums (R(f,P)) is the area under the curve calculated from multiple partition Partition P that is a set of x that has the same length between the closest one.

Supposing a function f(x);
with $\delta x_k = x_k - x_{k-1}$;
with P = { $x_0, x_1, x_2, x_3, \ldots$};
and a point ζ_k which is between x_k and x_{k-1}. (*this means that ζ_k doesn't have to be the same $\forall$ k*)

$$R(f,P) = \sum_{k=1}^{n} f(\zeta_k)\delta x_k$$

The Riemann sums is called the regular Riemann sums if δx_k have the same eqal length that is (b – a)/n for the Riemann sums on the interval [a,b]

Riemann Integration

Consider the Riemann sum with supremum and infimum of the f(x) instead of the $f(\zeta_k)$. (Become 2 sums, Upper sum and Lower sum)

To have Upper sum and Lower sum, the function f must be defined and bounded on a closed finite interval.

$$U(f,P) = \sum_{k=1}^{n} M_k\delta x_k$$

$$L(f,P) = \sum_{k=1}^{n} m_k\delta x_k$$

With M = sup f(x) and m = inf f(x)

As we are seeing the infimum and supremum, therefore it is the lowest and the highest bound, So we can bound the real Riemann sums by,

$$L(f,P) \leq R(f,P) \leq U(f,P)$$

If we add more partition, we can see that the lower sums will be greater while the upper sums will be smaller. Therefore, as the number of partition goes larger and larger, L(f,P) = R(f,P) = U(f,P).

Darboux Integral

There are two Darboux Integral, The Lower Darboux Integral and Upper Darboux Integral.

Lower Darboux Integral

The Lower sum of Riemann Sum with all possible partitions

Denoted by $\underline{\int_a^b} f(x)dx$

Upper Darboux Integral

The Upper sum of Riemann Sum with all possible partitions

Denoted by $\overline{\int_a^b} f(x)dx$

Important Notes

(a) Lower and Upper Darboux Integral exists $\forall$ bounded functions on all finite intervals
(b) $\underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx$

If the Upper and Lower Darboux Integral is equal, than the function f(x) is called *Riemann Integrable* on the Interval.

If f is Riemann Integrable on [a,b] and the fact that m and M is infimum and supremum of f respectively, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$