Concave and Convex Functions

- The function $f(x, y)$ is concave if its domain is convex and the line segment joining any two points on the graph is never above the graph
- Definition of a concave function
  - A function $f(\mathbb{x}) = f(x_1, \ldots, x_n)$ defined on a convex set S is concave in S if:
    - $f\big((1-\lambda)\mathbb{x}^0 + \lambda\mathbb{x}\big) \geq (1-\lambda)f(\mathbb{x}^0) + \lambda f(\mathbb{x})$
    - For all $\mathbb{x}^0, \mathbb{x} \in S$ and all $\lambda \in (0, 1)$
    - Eg. the line segment joining any two points in the domain is below the graph
- The function f is convex in S if $-f$ is concave
- Other definitions:
  - $f\ is\ concave\ iff\ M_f = \{(x, y): x \in S\ and\ y \leq f(x))\}$ is convex
  - $f\ is\ convex\ iff\ M_f = \{(x, y): x \in S\ and\ y \geq f(x))\}$ is convex
- Jensen's Inequality
  - A function f of n variables is concave on a convex set S in $\mathbb{R}^n$ iff the following inequality is satisfied for all $x_1, \ldots, x_n$ in S and all $\lambda_1 \geq 0, \ldots, \lambda_m \geq 0$ with $\lambda_1 + \cdots \lambda_m = 1$:
    - $f(\lambda_1\mathbb{x}_1 + \cdots + \lambda_m\mathbb{x}_m) \geq \lambda_1 f(\mathbb{x}_1) + \cdots + \lambda_m f(\mathbb{x}_m)$
  - Continuous version
    - Let $x(t)$ and $\lambda(t)$ be continuous functions in the interval $[a, b]$ with $\lambda(t) \geq 0$ and $\int_a^b \lambda(t)dt = 1$. If f is a concave function defined on the range of $x(t)$, then:
      - $f(\int_a^b \lambda(t)x(t)dt) \geq \int_a^b \lambda(t)f\big(x(t)\big)dt$

Useful Conditions for Concavity and Convexity

- f and g concave (convex) and $a \geq 0, b \geq 0$, then $af + bg$ concave (convex)
- $f(\mathbb{x})$ concave and $F(u)$ concave and increasing, then $U(\mathbb{x}) = F\big(f(\mathbb{x})\big)$ concave
- $f(\mathbb{x})$ convex and $F(u)$ convex and increasing, then $U(\mathbb{x}) = F(f(\mathbb{x}))$ convex
- f and g concave (convex), then $h(\mathbb{x}) = \min\{f(\mathbb{x}), g(\mathbb{x})\}\ is\ concave$ (convex)
- Suppose that $f(\mathbb{x}) = f(x_1, \ldots, x_n)$ has continuous partial derivatives in an open, convex set S in $\mathbb{R}^n$ then
  - f is concave in S iff for all $\mathbb{x}^0, \mathbb{x} \in S$
    - $f(\mathbb{x}) - f(\mathbb{x}^0) \leq \Sigma_{i=1}^n \frac{df(\mathbb{x}^0)}{dx_i}(x_i - x_i^0)$
  - f is strictly concave iff the inequality above is strict for all $\mathbb{x} \neq \mathbb{x}^0$
  - The corresponding result for convex (or strictly convex) functions is obtained by replacing $\leq$ (or $<$) by $\geq$ (or $>$) in the inequality above

Second-Derivative Tests for Concavity/Convexity: The two-variable case

- Let $z = f(x, y)$ be a function with continuous partial derivatives of the first and second order, defined over an open convex set S in the plane, then
  - f is concave iff:
    - $f''_{11} \leq 0, f''_{22} \leq 0$ and $\begin{vmatrix} f''_{11} & f''_{12} \\ f''_{21} & f''_{22} \end{vmatrix} \geq 0$
  - f is convex
    - $f''_{11} \geq 0, f''_{22} \geq 0$ and $\begin{vmatrix} f''_{11} & f''_{12} \\ f''_{21} & f''_{22} \end{vmatrix} \geq 0$
- Also possible to vary this to give sufficient conditions for strict concavity/convexity
  - f is strictly concave iff
    - $f''_{11} < 0$ and $\begin{vmatrix} f''_{11} & f''_{12} \\ f''_{21} & f''_{22} \end{vmatrix} > 0$

- f is strictly convex iff
    - $f''_{11} > 0$ and $\begin{vmatrix} f''_{11} & f''_{12} \\ f''_{21} & f''_{22} \end{vmatrix} > 0$
- Sufficient conditions for global extreme points
  - Let $f(x,y)$ be a function with continuous partial derivatives of the first and second order in a convex domain S, and let (x_0, y_0) ben an interior point of S and which f is stationary
    - If for all (x,y) in S, one has $f''_{11}(x,y) \leq 0, f''_{22}(x,y) \leq 0$ and $f''_{11}(x,y)f''_{22}(x,y) - [f''_{12}(x,y)]^2 \geq 0$, then (x_0, y_0) is a maximum point for $f(x,y)$ in S

Second-Derivative Tests for Concavity/Convexity: The n-variable case

- Suppose that $z = f(\mathbb{x})$ is a C^2 funciton in a domain S in $\mathbb{R}^n$
  - $\mathbb{H}(\mathbb{x}) = \left[f''_{ij}(\mathbb{x})\right]_{m \times n}$ is the Hessian matrix of
  - The n determinants:
    - $D_k(\mathbb{x}) = \begin{vmatrix} f''_{11}(\mathbb{x}) & \dots & f''_{1k}(\mathbb{x}) \\ \vdots & \ddots & \vdots \\ f''_{k1}(\mathbb{x}) & \dots & f''_{kk}(\mathbb{x}) \end{vmatrix} \; (k = 1, \dots, n)$
    - Are called the leading principal minors of $\mathbb{H}(\mathbb{x})$
- Then:
  - f is strictly concave in S iff
    - $(-1)^k D_k(\mathbb{x}) > 0$ for $k = 1, \dots, n$ and for all $\mathbb{x} \in S$
  - f is strictly convex in S iff
    - $D_k(\mathbb{x}) > 0$ for $k = 1, \dots, n$ and for all $\mathbb{x} \in S$
- Saddle point
  - If $D_n(\mathbb{x}^0) \neq 0$ and the function is neither concave or convex, then a stationary point $\mathbb{x}^0$ is a saddle point

Quasi-Concave and Quasi Convex Functions

- The function f, defined over a convex set $S \subset \mathbb{R}^n$, is quasi-concave if the upper level set $P_a = \{\mathbb{x} \in S : f(\mathbb{x}) \geq a\}$ is convex for each number a
  - Quasi convex if $-f$ is quasi-concave
    - f is quasi-convex iff the lower level set $P_a = \{\mathbb{x} : f(\mathbb{x}) \leq a\}$
  - Also:
    - If $f(\mathbb{x})$ is concave, then $f(\mathbb{x})$ is quasi-concave
    - If $f(\mathbb{x})$ is convex, then $f(\mathbb{x})$ is quasi-convex
- Let f be a function of n variables defined over a convex set S in $\mathbb{R}^n$, then f is quasi-concave iff, for all $\mathbb{x}, \mathbb{x}^0 \in S, and\ all\ \lambda \in \{0,1\}$, one has
  - $f(\mathbb{x}) \geq f(\mathbb{x}^0)$, then
    - $f(1-\lambda)\mathbb{x} + \lambda\mathbb{x}^0) \geq f(\mathbb{x}^0)$
- Properties
  - A sum of quasi-concave (quasi-convex) functions is not necessarily quasi-concave (quasi-convex)
  - If $f(\mathbb{x})$ is quasi-concave (quasi-convex) and F is strictly increasing, then $F\big(f(\mathbb{x})\big)$ is quasi-concave (quasi-convex)
  - If $f(\mathbb{x})$ is quasi-concave (quasi-convex) and F is strictly decreasing, then $F\big(f(\mathbb{x})\big)$ is quasi-convex (quasi-concave)
- Determinant condition
  - Bordered Hessians

    - Ordinary Hessians used for examining concavity of a function are 'bordered' by an extra row and column consisting of the first order-partials of the function:

$$D_r(\mathbf{x}) = \begin{vmatrix} 0 & f_1'(\mathbf{x}) & \cdots & f_r'(\mathbf{x}) \\ f_1'(\mathbf{x}) & f_{11}''(\mathbf{x}) & \cdots & f_{1r}''(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ f_r'(\mathbf{x}) & f_{r1}''(\mathbf{x}) & \cdots & f_{rr}''(\mathbf{x}) \end{vmatrix}$$

    - Necessary condition for f to be quasi-concave is that $(-1)^r D_r(\mathbb{x}) \geq 0$ for $r = 1, \ldots, n$ and all $\mathbb{x} \in S$
    - A sufficient condition for f to be quasi-concave is that $(-1)^r D_r(\mathbb{x}) > 0$ for $r = 1, \ldots, n$ and all $\mathbb{x} \in S$

Local Extreme Points

- Saddle point
  - A stationary point with the property that there exist points (x, y) arbitrarily close to (x_0, y_0) with $f(x, y) < f(x_0, y_0)$ and there also exists such points with $f(x, y) > f(x_0, y_0)$
- Second derivative test for local extrema
  - Suppose $f(x, y)$ is a C^2 function in a domain S, and let (x_0, y_0) be an interior stationary point of S
  - Let
    - $A = f_{11}''(x_0, y_0), B = f_{12}''(x_0, y_0), C = f_{22}''(x_0, y_0)$
  - Then:
    - If $A < 0$ and $AC - B^2 > 0$, then (x_0, y_0) is a (strict) local maximum point
    - If $A > 0$ and $AC - B^2 > 0$, then (x_0, y_0) is a (strict) local minimum point
    - If $AC - B^2 < 0$, then (x_0, y_0) is a saddle point
    - If $AC - B^2 = 0$, then (x_0, y_0) could be a local maximum, local minimum or a saddle point
  - Note that if $AC - B^2 > 0$, then $AC > 0$ and if $A > 0$, then $C > 0$ by implication

Extreme Value Theorem

- Sets
  - A point (a, b) is called an interior point of a set S in the plane if there exists a circle centred at (a, b) such that all points strictly inside the circle lie in S
    - A set is an open set if it consists only of interior points
  - The point (a, b) is called a boundary point of a set S is every circle centred at (a, b) contains points of S as well as points in its complement
    - If S contains all its boundary points, it is closed
  - Bounded
    - If the whole set is contained within a sufficiently large circle
  - Compact 11
    - A set that is both closed and bounded
- Extreme value theorem
  - Suppose the function $f(x, y)$ is continuous throughout a nonempty, closed and bounded set S in the plane

    - Then there exists both a point (a, b) in S where f has a minimum and a point (c, d) in S where f has a maximum:
        - $f(a, b) \leq f(x, y) \leq f(c, d)$ for all (x, y) in S
- Finding maxima and minima
    - Procedure
        - Find all stationary points of f in the interior of S
        - Find the largest and the smallest value of f on the boundary of S, along witht eh associated points
        - Compute the values of the function at all the points found in step 1 and 2. The largest function value is the maximum and the smallest is the minimum