STAT2012 LECTURE NOTES

STATISTICAL TEST (ADVANCED)

Contents

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Lecture 1. Monday, 25 July 2016

Course summary

1 quiz 14 Sep (5)

2 assignments (10) (Aug 24, Oct 12)

Weekly computer labs (10%)

1 hour comp exam (open book) (10) (26 Oct)

2 Hour final (65%)

HAND IN COMP LABS BY 5PM THE DAY OF THE LAB

Introduction

Assumptions: sample $X_1, X_2, \dots, X_N$

Hypotheses: H vs H_A

Test statistics: $T = T(X_1, X_2, \dots X_n)$

$p -$ value: $p = P_H(T \geq t)$

Conclusion: by "interpreting" the p value

- Often come across data in various sizes/formats. Data is the raw material for statistics

Example 1.1 Beer labelling

A brand claims beer content is 375 mL on the label. A sample of 40 bottles gives a sample average of 373.9 and a standard deviation of 2.5

- Is the labelling correct?

Eg 1.2: Height comparison

Is there evidence that girls are taller than boys on their 10th birthday?

- The nature of data is different to the first one; as there are 2 samples here (boys and girls) compared to just the beer bottles

Eg 1.3: Temperature

With a bunch of observed temperatures; is there evidence it is NOT normally distributed?

Aim of course:

- Attempt to answer these yes/no questions. Introduce methods of applied stats including parametric and non parametric; for the analysis of data and techniques of statistical inference for 1,2 or several samples

Methodology of Statistical testing

Basic concepts:

Population

Collection of all possible results that are taget of interest, described by random variable X

Sample

Is a subset of the population; $X_1, \ldots, X_n$

- We require that each element in a dataset is **representative** of the population; this is equivalenent to saying that $X_1, \ldots, X_n$ are required to be iid RVs

Dataset:

$x_1, \ldots, x_n$ is an observed value of the sample $X_1, \ldots, X_n$. Sometimes also sample or sample value

Steps of testing:

1. Model and assumptions

The model represents our belief about the probability distribution F describing the population; with $X \sim F_\theta(x)$ where $F_\theta(x)$ is a specific distribution function only depending on the unknown parameter θ .

Eg: $X \sim N(\mu, \sigma^2); X \sim B(n, p)$ wth parameters μ, σ^2, p these corresponding statistical analysis therefore are called **parametric**

A statistical analysis is called **nonparametric** if our assumption on the population distribution F do not specify a particular class of probability distribution.

Parametric hypothesis testing:

$X_1, \ldots, X_N$ ind RV with $X_i \sim F_\theta(x) \forall i \in (1, n)$

Non parametric hypothesis testing

$X_1, \ldots X_n$ are iid random variables with distribution function F

Note: in general we also require F to not be heavy (long)-tailed; require certain moment conditions. When the F is heav tailed, the boxplot of the dataset should be skewed. For skewed data, we can sometimes transform them into symmetric. (Discussed later)

2. Hypothesis on (the parameter in) the population

Null hypothesis

The statement against which we are searching for evidence is called the <u>null hypothesis</u>, denoted H_0. The null hypothesis is often a “no difference” statement.

Alternative hypothesis

The statement we will consider if H_0 is false is called the alternative hypothesis, H_1

Eg: if $X \sim F_\theta(x)$ we may have

$$H_0:\ \theta = \theta_0 \ \text{(against)}$$

$$H_1:\ \theta > \theta_0 \text{ (right sided alternative)}$$

$$\theta < \theta_0 \text{ (left sided alternative)}$$

$$\theta \neq \theta_0 \text{ (two sided alternative)}$$

Example 1.1 (cont) Beer labels

Assume claim amount from $X \sim N(\mu, \sigma^2)$ then null hypothesis is:

$$H_0: \mu = 375$$
$$H_1:\ \mu < 375 \ \text{(as seeing if less beer than labelled)}$$

Eg 1.2: children height

Assume girls $X \sim N(\mu_1, \sigma_1^2)$ and boys $Y \sim N(\mu_2, \sigma_2^2)$ then null hypothesis $H_0: \mu_1 - \mu_2 = 0; H_1: \mu_1 - \mu_2 > 0$

Eg 1.3 Temperature normally distributed

$H_0: X \sim N(\mu, \sigma^2); H_1: X \nsim\ N$

3. Finding the test statistics, T

A function $T := T(X_1, \dots, X_N)$ of the sample $X_1, \dots, X_n$ is called a test statistics. To be a useful statistics for testing H_0, T is chosen such that:

- The distribution of T is completely determined when H_0 is true; $\theta \in \Theta_0$
- The particular observed values of T, called rejection region, can be taken as evidence of poor agreement with the assumption that H_0 is true.

A **test statistic** is a statistic (a quantity derived from the sample) used in statistical hypothesis testing.[1] A hypothesis test is typically specified in terms of a test statistic, considered as a numerical summary of a data-set that reduces the data to one value that can be used to perform the hypothesis test. In general, a test statistic is selected or defined in such a way as to quantify, within observed data, behaviours that would distinguish the null from the alternative hypothesis, where such an alternative is prescribed, or that would characterize the null hypothesis if there is no explicitly stated alternative hypothesis.

4. Calculate the $p-$value (observed significance level)

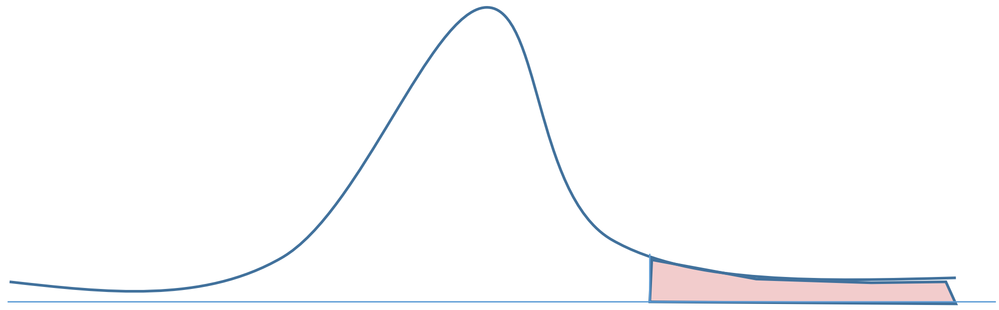
For a given data $x_1, \dots x_n$ we can calculate $t_{obs} = T(x_1, \dots, x_n)$, the observed value of the test statistic T.

The p value, or observed significance level, is defined formally as the smallest α level, such that its corresponding rejection region R contains t_{obs}

Eg:

$$H_0: \mu = \mu_0$$
$$H_1: \mu < \mu_0$$

Test statistic: $t = 2.594$ with $n = 7$



t statistic

Eg: assume large values of T support the alternative, that is: R has the form $[r, \infty)$ where r can be calculated

- The smaller the p value is equal to the probability under H_0 that the test statistica T is as extreme (or more) as t_{obs} ; quantifies evidence against H_0 in the following sense:
    - If p is small, then either H_0 is true and poor agreement due to an unlikely event; or H_0 is false, therefore **the smaller the p value, the stronger the evidence against H_0 in favour of H_1**
    - A larger p value does not mean that there is evidence H_0 is true, only that th test detects no inconsistency between predictions of H_0 and true experiment

5. Conclusion

The p value is a probability, $p \in [0,1]$. We can assess the significance of the evidence provided by the data against H_0 by interpreting the $p -$ value.

Although such rules are arbitrary, we use the following convention in this course:

$p > 0.1$ data consistent with H_0

$p \in [0.05,0.1)$ borderline against H_0

$p \in [0.025,0.05)$ reasonable strong evidence against H_0

$p \in [0.01,0.025)$ strong evidence against H_0

$p < 0.01$ very strong evidence against H_0

Review of important results:

Sample mean:

$$\bar{X} = \frac{1}{n}\sum_{n=1}^{n} X_i$$

Sample variance:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

For $X_1, \dots X_n$ iid $N(\mu, \sigma^2)$ Random variables:

For $X_1, \dots, X_n \sim N(\mu, \sigma^2)$ iid continuous RV's:

$$\frac{\bar{X} - \mu}{\left(\frac{\sigma}{\sqrt{n}}\right)} \sim N(0,1)$$

$$\frac{\sum_{i=1}^{n}(X_i - \mu)^2}{\sigma^2} \sim \chi_n^2$$

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$$

$$\frac{\bar{X} - \mu}{\left(\frac{S}{\sqrt{n}}\right)} \sim t_{n-1}$$

$\bar{X}$ and S^2 are indepdent random variables

Useful R commands:

pnorm(x,m,s) $pP(X \leq x)$ with $X|sim\ N(m, s^2)$

pt(x,d) $= P(X \leq x); X \sim t_d$

pchisq(x,d) $= P(X \leq x); X \sim \chi_d^2$

qnorm(p,m,s) is a number x such that $P(X \leq x) = p$ where $X \sim N(m, s^2)$

Lecture 2. Tuesday, 26 July 2016

Tests for Population mean:

- Normal population with unknown variance
- One sample and paired sample
- One sample t test and paired sample t test

Reminder of yesterday:

1. Assumptions: sample $X_1, \dots, X_n$
2. Hypotheses $H_0\ \ vs\ H_A$
3. Test statistics $T = T(X_1, \dots, X_n)$
4. p value
5. $conclusion$

Normal population

One sample: suppose we have a sample $(X_1, \ldots, X_n)$ of the size n drawn from a normal population with unkown variance. We want to make some statement about the population mean μ.

$t-$ test for one sample

Assumptions

X_j: are iid random variables with $X_j \sim N(\mu, \sigma^2)$ where σ^2 is unkown.

Hypothesis:

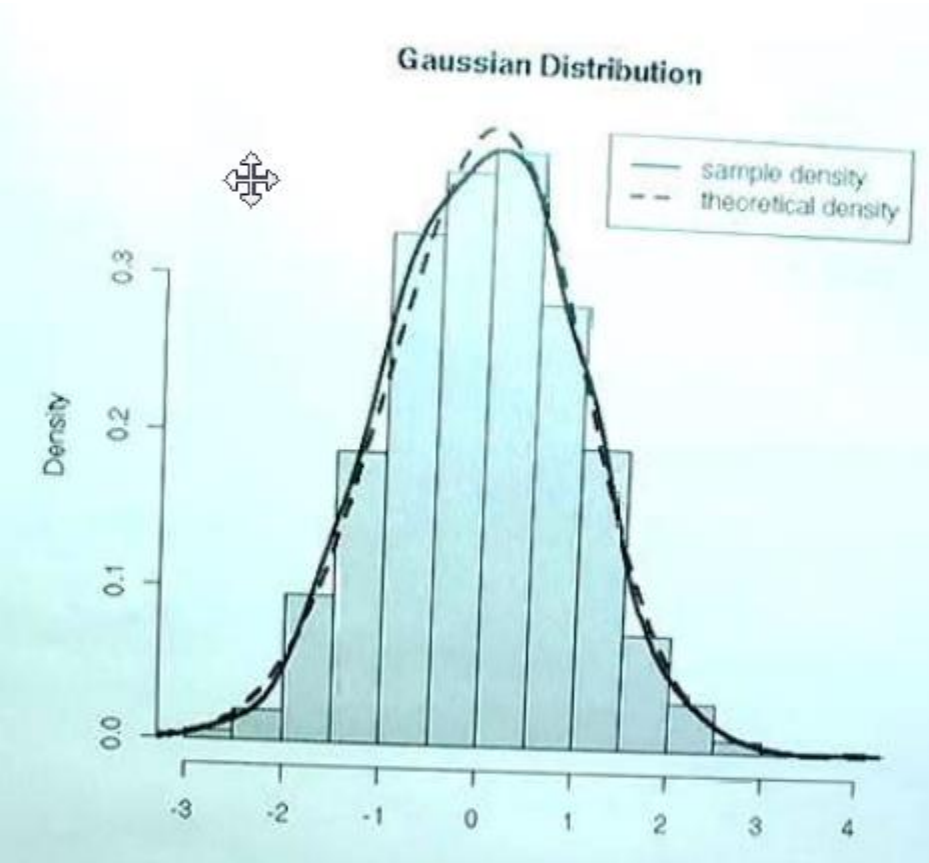
$$H: \mu = \mu_0$$

Vs

$$H_A: \ \mu > \mu_0 \text{ or}$$

$$\mu < \mu_0 \text{ or}$$

$$\mu \neq \mu_0$$



Test statistics (Studen'ts $t-$ statistics)

$$T_n = \frac{\sqrt{n}(\bar{X} - \mu_0)}{S}$$

Where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$

Note that $T_n \sim t_{n-1}$ under $H: \mu = \mu_0$

p value
Let $(x_1, x_2, \ldots, x_n)$ be a dataset and t be the observed value of T_n; ie.

$$t = \frac{\sqrt{n}(\bar{x} - \mu_0)}{s}$$

Where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$

We then want to find the p value as:

$$p = P(t_{n-1} \geq t) \text{ for } H_A: \mu > \mu_0$$

$$p = P(t_{n-1} \leq t) \text{ for } H_A: \mu < \mu_0$$

$$p = 2P(t_{n-1} \geq |t|) \text{ for } H_A: \mu \neq \mu_0$$

Decision:
We want to interpret our p value.

$$p > 0.1 \text{ data consistent with } H_0$$

$$p \in [0.05, 0.1) \text{ borderline against } H_0$$

$$p \in [0.025, 0.05) \text{ reasonable strong evidence against } H_0$$

$$p \in [0.01, 0.025) \text{ strong evidence against } H_0$$

$$p < 0.01 \text{ very strong evidence against } H_0$$

The p value is the probability; so it is between $[0,1]$. We can assess the significance of the evidence provided by the data against the null hypothesis by "interpreting" the p value.

Although such rules are arbitrary, in this course we shall use:

P values for STAT2912:
$p > 0.1$ Data consistent with H_0

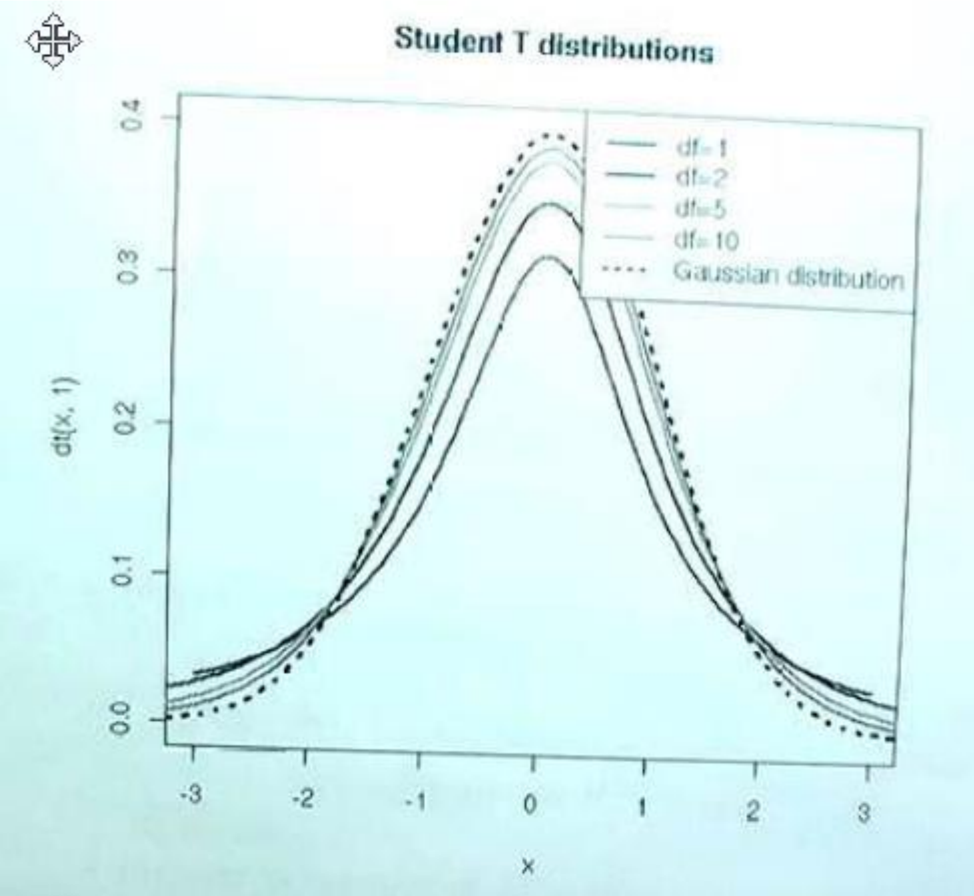
$p \in [0.05, 0.1)$ Borderline evidence against H_0

$p \in [0.025, 0.05)$ Reasoable strong evidence against H_0

$p \in [0.01, 0.025)$ Strong evidence against H_0

$p < 0.01$ Very strong evidence against H_0

Student's T distribution



- Has a wider tail than the normal; so used if we are not sure if the sample is big enough.

Remarks on t test:

We have $T_n \sim t_{n-1}$ under $H_0: \mu = \mu_0$

$\bar{X}$ is an ideal estimator of μ, which means that $\bar{X}$ is near μ_0 if $H: \mu = \mu_0$ is true. Therefore it is reasonable to believe that we should reject H in favour of H_A if $t = \frac{\sqrt{n}(\bar{x}-\mu_0)}{s}$ (observed value of T_n)) is large (negative large or both depending on H_A) . On the other hand, if t is large, then p valus is small. This coincided with our 'interpretation' of the p value.

Note that $P(t_{n-1} \geq t) = P(t_{n-1} \leq -t)$ for $t \geq 0$. And that $P(|t_{n-1}| \geq |t|) = 2P(t_{n-1} \geq |t|)$.

In R:

pt(t,n-1) gives $P(t_{n-1} \leq t)$

Examples:

Example (beer)

Beer contents in a six pack is …. ; is the mean content of the beer less that 375 as labelled?

Solution:

Consider sample $X_1, \dots, X_6$ normally distributed; $X_j \sim N(\mu, \sigma^2)$ with σ^2 unkown

Hypothesis:

$$\mu = 375$$
$$H_1: \mu < 375$$

For the data:

$$\bar{x} = 374.87; s^2 = 0.087$$

And the observed value of the test statistics is:

$$T_n = \frac{\sqrt{n}(\bar{X} - 375)}{s} = \frac{\sqrt{6}(374.87 - 375)}{0.087} = -1.1094$$

The p value is:

$p = P(t_5 \leq t) = 0.1589$ using the table of pt(t,5).

So the data is consistent with the claim on the label; as $p > 0.1$

In R:
t.test(x,alternateive = ??, mu=μ_0)

is used to construe the one-saple t test, where x is the data, ?? can be less, greater or two.sided depeding on what you want

Remarks:

- For small sample size, $(n \leq 20)$ the t test is sensitive to the assumption that the sample is from a normal population. It is recommended that all data be checked for shape by looking at a boxplot and a normal qq-plot of the data (in R: boxplot and qqnorm command)
- For small sample size $(n \leq 20)$ it is required that the boxplot of the data should be as symmetric as possible and the points in qqplot be nearly around a straight line. (note boxplot and qqplot may be misguided if n is very small $(n \leq 10)$).

- If the sample size is large $(n \geq 20)$; as long as the data does not come from a long-tail distribution, the t test should be fine. (discussed lecture 3)

Paired sample

Suppose we have a sample of paired observations (eg: before/after data or twin data). We want to make some statement about the mean difference.

Example 2/2 (Blood samples from smokers)
Blood samples from indivisuals before/after smoking are used to measure aggregation of blood platelets: …..

Question: is the aggregations affected by smoking?

Analysis of paired tests:

Let X_i and Y_i be th blood platelets before/after be each individual. We get a sample of paired observations $(x_1, y_1), \dots, (x_n, y_n)$

(note that X, Y are dependent). However, we want to know the difference)

$$d_i = X_i - Y_i$$

As d is from different individuals, we can assume that d_i is ind RV's with $d_j \sim N(\mu, \sigma^2)$ with unkown σ^2

And answer the question by testing the following:

$$H: \mu = 0; \;\; H_z: \mu \neq 0$$

This reduces the paired sample problem to a single sample (of a difference),, with testing $\mu = 0$

In general;

Let $(x_1, y_1), \dots (x_n, y_n)$ be a sample of paird observations. We wish to test

$$H: \mu_x = \mu_y$$

Vs

$$H_A: \mu_x >, <, \neq \mu_y$$

Which is the same as $d = X_j - Y_j$

$$H: \mu_d = 0 \;\; vs \; H_A \; \mu_d, >, <, \neq 0$$

Paired sample t test:

Assumptions:

$d_i = X_i - Y_i$ are a rand samp from a normal pop with unkown σ^2; that is $d_i \sim N(\mu, \sigma^2)$

Test statistics:

$T_n = \frac{\sqrt{n}\bar{d}}{s_d} \sim t_{n-1}$ under H

P value:

Same as before

In R:

```
t.test(x,y,alternative ??, mu=9,paired = TRUE)
```

Eg: 2.2

We have

Before: 25 25 27 44 30 67 53 53 52 60 28;
After: 27 29 37 36 46 82 57 80 61 59 43.

x<-c(25, 25, 27, 44, 30, 67, 53, 53, 52, 60, 28) > y<-c(27, 29, 37, 36, 46, 82, 57, 80, 61, 59, 43) >

t.test(x,y, alternative="two.sided", mu=0, paired=T)

Paired t-Test data: x and y t = -2.9065, df = 10, p-value = 0.0157

alternative hypothesis: true mean of differences is not equal to 0 95 percent confidence interval: -14.93577 -1.97332

sample estimates: mean of x - y -8.454545

the p value value is 0.0156. hence strong evidence against H_0 (so strong evidence people affected by smoking)

RemarksL

- Recommended that the data difference shape chaecked in a boxplot/ normal qqplot
- For small sample, boxplot should be as symmetric as possible; and qqplot as near a straight line.
- For large sample a t test is fine as long as it is not from a long tail distribution

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Lecture 3. Wednesday, 27 July 2016

Large sample size tests for population mean

If sample size is large ($n \geq 20$)

Test for population mean:

Suppose we want to test a population mean μ based on a random sample $X_1, \ldots, X_n$; where the sample size n is considered to be large enough ($n \geq 20$)

Let $(x_1, x_2, \ldots, x_n)$ be an observed value

If the sample does not come from a heavy tail distribution (too many outliers in boxplot), we may use the following procedures to test hypothesis about population mean μ.

Common large sample tests:

Hypothesis:

$$H: \mu = \mu_0; vs\ H_A: \mu >, <, \neq \mu_0$$

Test statistics:

Students t statistic: $T_n = \frac{\sqrt{n}(\bar{X}-\mu_0)}{S}$where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i\,; S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$

(note that we do not have $T_n \sim t_{n-1}$)

Or Z statistic: (given population variance σ^2) $Z_n = \frac{\sqrt{n}(\bar{X}-\mu_0)}{\sigma}$ where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

P value

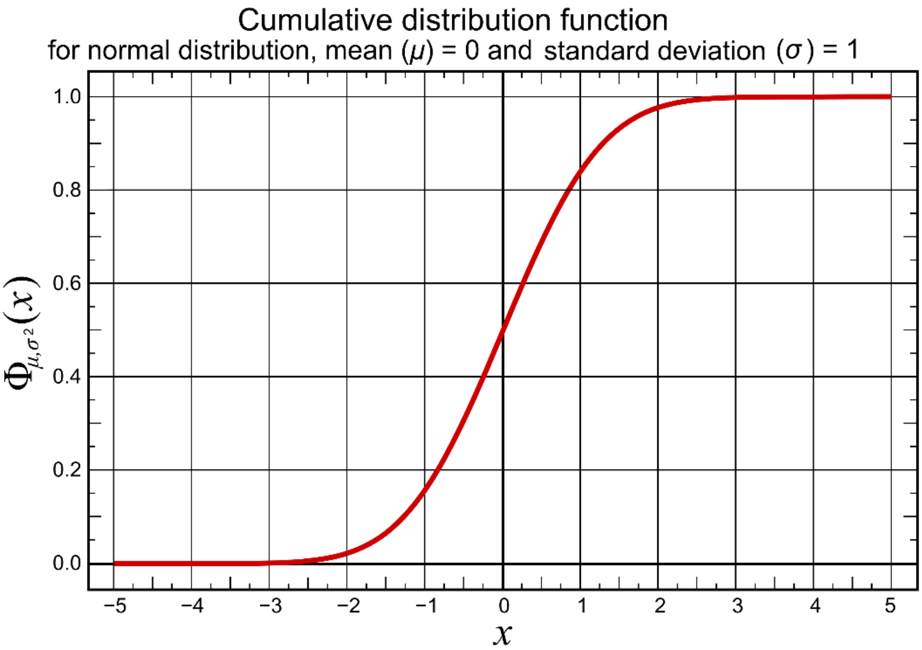
Let t be the observed value of T_n or Z_n, that is

$$t = \frac{\sqrt{n}(\bar{x}-\mu_0)}{s} \quad or$$
$$t = \frac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma} \quad (with\ given\ \sigma)$$

If the sample size n is large enough, the p value is approximately given by

$$p \approx 1 - \Phi(t) \quad for\ H_A: \mu > \mu_0$$
$$p \approx \Phi(t) \quad for\ H_A: \mu < \mu_0$$
$$p \approx 2\big(1 - \Phi(|t|)\big)\ for\ H_A: \mu \neq \mu_0$$

Where $\Phi(t)$ is the CDF of the normal $N(0,1)$ distribution.



Remarks:

- The paired sample may be discussed in a similar manner
- The choice of test statistic is based on the fact that $\bar{X}$ is a point estimate of μ. It means that $\bar{X}$ is close to μ_0 if $H: \mu = \mu_0$ is true. If in reality, $\mu \neq \mu_0$, $|\bar{X} - \mu_0|$ is more likely to be large.
- If the sample is from a normal population with given variance σ^2, the test procedure for μ (with statistic Z_n) is still applicable for small sample size ($n \leq 20$). Note that under $H: \mu = \mu_0$; $Z_n = \frac{\sqrt{n}(\bar{X}-\mu_0)}{\sigma} \sim N(0,1)$ for each $n \geq 1$ in this case.
- Calculation of the p value is based on the CENTRAL LIMIT THEOREM
- Large sample size for μ is equivalen to t test if the sample size is large enough.

- Note: for large sample size, we don't need to check the box plot/qq plot to check for normality

Example 3.1: (vice president)

A VP of sales claims that people are averagine no more that 15 sales each week. He samples 24 people, and the number of contacts they made recorded.

18, 17, 18, 13, 15, 16, 21, 14, 24, 12, 19, 18,
17, 16, 15, 14, 17, 18, 19, 20, 13, 14, 12, 15

$$H_0: \mu \leq 15; \;\; H_A: \mu > 15$$

We get that:

$$t = \frac{\sqrt{24}(\bar{x} - 15)}{s}$$
$$p = 1 - \Phi(t)$$

In R:

```
> x<-c(18,17,18,13,15,16,21,14,24,12,19,18, 17,16,15,14,17,18,19,20,13,14,12,15)
> barx<-mean(x) >
 s<-sqrt(var(x))
> t<-sqrt(24)*(barx - 15)/s
 > p<-1-pnorm(t)
> p
 [1] 0.007954637
```

As $p < 0.01$ there is strong evidence to indicate the VP is incorrect, that is; the average number of sales contact per week exceeds 15.

We can also do the one sample t test command in R:

```
> x<-c(18,17,18,13,15,16,21,14,24,12,19,18, 17,16,15,14,17,18,19,20,13,14,12,15)
> t.test(x, alternative="greater",mu=15)
One-sample t-Test
data: x t = 2.411, df = 23, p-value = 0.0121
alternative hypothesis: true mean is greater than 15 95 percent confidence interval: 15.42167 NA
sample estimates: mean of x 16.45833
```